A Game-Theoretic Decision Procedure for the Constructive Description Logic $c\mathcal{ALC}$

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Abstract

In recent years, several languages of non-classical description logics have been introduced to model knowledge and perform inference on it. There have been several proposals for different application scenarios. The *constructive* description logic $c\mathcal{ALC}$ deals with uncertain and dynamic knowledge and is therefore more restrictive than intuitionistic $\mathcal{ALC}$.

We make use of a game-theoretic dialogue-based proof technique that has its roots in philosophy to explain reasoning in $c\mathcal{ALC}$ and its modal-logical counterpart CK_n.

The game-theoretic approach we build on has been introduced by Kuno Lorenz and Paul Lorenzen. It contains a selection of *game rules* that specify the behaviour of the proof system. Other logics can be adapted or even constructed by changing the underlying game rules.

We formalize the intuitive rules in first-order logic making them concrete from a mathematical perspective and thereby provide an adequate semantics for $c\mathcal{ALC}$. It turns out that the interaction semantics provides the right level of constructiveness.

This report contains only the soundness proof for a modal dialogical system presented in the article *A Game-Theoretic Interpretation for the Constructive Modal Logic CK.*

1 Frame Rules Formalized

1.1 Valid Moves (frame-independent)

1. **Well-Formed Play (F1a)**

$$\begin{aligned} &\mathit{well\_formed}((\mathit{moves}, \mathfrak{f}), m) \Leftrightarrow \\ &\qquad (\ \mathit{id}(m) = \mathit{id}(\mathit{last}(\mathit{moves})) + 1 \ \wedge \\ &\qquad\quad \mathit{well\_formed\_play}((\mathit{moves} \cup \{m\}, \mathfrak{f})) \ ). \end{aligned}$$

2. **Particle Assignment (F1b)**

$$
\begin{aligned}
& \mathit{particle\_assign}((\mathit{moves}, \mathfrak{f}), m) \ \Leftrightarrow \\
& \quad (\ \exists m' \in \mathit{moves} \ (\\
& \quad\quad m' \overset{g}{\rightarrowtail} m \ \wedge \ (\mathit{assert}(m'), \mathit{assert}(m)) \in \mathit{PRuleSet} \ \ ) \ ) \\
& \quad \text{where } g = (\mathit{moves} \cup \{m\}, \mathfrak{f})
\end{aligned}
$$

3. **O's Atom Attacks (F1c)**

$$
\begin{aligned}
& \mathit{atom\_attack\_O}((\mathit{moves}, \mathfrak{f}), m) \ \Leftrightarrow \ \forall \phi, f, e \quad (\\
& \quad (\quad (\ \mathit{assert}(m) = (\mathsf{O}, f, e) \ \wedge \ \exists m' \in \mathit{moves} \ (\\
& \quad\quad\quad m' \overset{g}{\rightarrowtail} m \ \wedge \ \mathit{expr}(m') = \phi \ \wedge \ \mathit{atom}(\phi) \ ) \ ) \\
& \quad\quad \Rightarrow \ (\ e = \lhd \ \ \wedge \ \ f = \phi\mathbf{?} \ ) \quad) \qquad \wedge \\
& \quad (\quad (\ \mathit{assert}(m) = (\mathsf{P}, f, e) \ \wedge \ \exists m' \in \mathit{moves} \ (\\
& \quad\quad\quad m' \overset{g}{\rightarrowtail} m \ \wedge \ \mathit{assert}(m') = (\mathsf{O}, \phi\mathbf{?}, \lhd) \ \wedge \ \mathit{wrld}(m') = w \ \wedge \ \mathit{atom}(\phi) \ ) \ ) \\
& \quad\quad \Rightarrow \ (\ e = \lhd \ \ \wedge \ \ f = \mathbf{!} \ \wedge \ \exists m^* \in \mathit{moves} \ (\\
& \quad\quad\quad \mathit{player}(m^*) = \mathsf{O} \ \wedge \ \mathit{expr}(m^*) = \phi \ \wedge \ \mathit{wrld}(m^*) = w \ ) \ ) \ ) \quad) \\
& \quad \text{where } g = (\mathit{moves} \cup \{m\}, \mathfrak{f})
\end{aligned}
$$

4. **General Repetition Restriction (F1d)**

$$
\begin{aligned}
& \mathit{repet\_a}((\mathit{moves}, \mathfrak{f}), m) \ \Leftrightarrow \ \forall n, f, \phi, w, \rho \\
& \quad (\quad m = (n, (\mathsf{P}, f, \phi), w, \rho) \ \Rightarrow \ \neg \exists m' \in \mathit{moves} \ (\\
& \quad\quad \exists n' \ (m' = (n', (\mathsf{P}, f, \phi), w, \rho) \ \wedge \ n \neq n' \ \ ) \quad) \ )
\end{aligned}
$$

$$
\begin{aligned}
& \mathit{repet\_b}((\mathit{moves}, \mathfrak{f}), m) \ \Leftrightarrow \ \forall n, n', \rho \\
& \quad (\quad \mathit{player}(m) = \mathsf{O} \quad \Rightarrow \\
& \quad\quad \neg \exists m' \in \mathit{moves} \ (\ \mathit{player}(m') = \mathsf{O} \ \wedge \ \mathit{id}(m) \neq \mathit{id}(m') \ \wedge \ \mathit{ref}(m) = \mathit{ref}(m') \ ))
\end{aligned}
$$

5. **Intuitionist Defence Restriction (F1e)**

$$
\begin{aligned}
&\mathit{defence\_i}((\mathit{moves}, \mathfrak{f}), m) \;\Leftrightarrow\; \forall m' \in \mathit{moves} \\
&\quad (\quad (\quad \mathit{defence}(m) \;\wedge\; m' \overset{g}{\rightarrowtail} m \quad) \\
&\qquad \Rightarrow \;\neg\exists m^* \in \mathit{moves} \; (\\
&\qquad\qquad \mathit{player}(m') = \mathit{player}(m^*) \;\wedge\; \mathit{attack}(m^*) \;\wedge \\
&\qquad\qquad \mathit{id}(m) > \mathit{id}(m^*) > \mathit{id}(m') \quad) \quad) \\
&\quad \text{where } g = (\mathit{moves} \cup \{m\}, \mathfrak{f})
\end{aligned}
$$

6. **P's Atom Attacks (intuitionist) (F1f)**

$$
\begin{aligned}
&\mathit{atom\_attack\_P\_i}((\mathit{moves}, \mathfrak{f}), m) \;\Leftrightarrow\; \forall n, f, e, \phi, w, \rho \; (\\
&\qquad (\; m = (n, (\mathsf{P}, f, e), w, \rho) \;\wedge\; \exists m' \in \mathit{moves} \; (\\
&\qquad\qquad m' \overset{g}{\rightarrowtail} m \;\wedge\; \mathit{expr}(m') = \phi \;\wedge\; \mathit{atom}(\phi) \;) \;) \\
&\qquad \Rightarrow \; \exists m^* \in \mathit{moves} \;\; (\; \exists n^*, \rho^* \;\; (\\
&\qquad\qquad m^* = (n^*, (\mathsf{P}, f, e), w, \rho^*) \;\wedge\; \mathit{attack}(m^*) \;\wedge\; n^* < n \quad) \quad) \;) \\
&\quad \text{where } g = (\mathit{moves} \cup \{m\}, \mathfrak{f})
\end{aligned}
$$

7. Summary

$$
\begin{aligned}
&\forall g \forall m (\mathit{valid\_move}(g, m) \Leftrightarrow \\
&\qquad (\quad \mathit{well\_formed}(g, m) \;\wedge\; \mathit{particle\_assign}(g, m) \;\wedge \\
&\qquad\quad \mathit{atom\_attack\_O}(g, m) \;\wedge\; \mathit{repet\_a}(g, m) \;\wedge\; \mathit{repet\_b}(g, m) \\
&\qquad\quad \mathit{defence\_i}(g, m) \;\wedge\; \mathit{atom\_attack\_P\_i}(g, m) \quad)
\end{aligned}
$$

1.2 Frame Updates

1. **(F2a)**

$$
\begin{aligned}
&\mathit{frame\_update\_1}((\mathit{moves}, \mathfrak{f}), m^*, \mathfrak{f}^*) \Leftrightarrow \\
&\qquad (\mathit{player}(m^*) = \mathsf{P} \;\Rightarrow\; \mathfrak{f}^* = \mathfrak{f} \;) \quad)
\end{aligned}
$$

2. **(F2b)**

$$\begin{array}{l} frame\_update\_2a((moves,(W,\longrightarrow)),m^*,(W^*,\longrightarrow^*)) \;\Leftrightarrow\; \forall w',w^*,i,\varphi \\ \quad (\;(\;defence(m^*) \;\wedge\; w^* = wrld(m^*) \;\wedge\; \exists m' \in moves\;(\\ \qquad m' \overset{g}{\rightarrowtail} m^* \;\wedge\; force(m') = \varphi\mathbf{?}\Diamond_i \;\wedge\; w' = wrld(m')\;)\;) \\ \quad \Rightarrow\; (\;W^* = W \cup \{w^*\} \;\wedge\; \longrightarrow^* = \longrightarrow \cup \{(i,w',w^*)\}\;)\quad) \\ \quad \text{where } g = (moves \cup \{m^*\},(W,\longrightarrow)) \end{array}$$

$$\begin{array}{l} frame\_update\_2b((moves,(W,\longrightarrow)),m^*,(W^*,\longrightarrow^*)) \;\Leftrightarrow\; \forall w^*,w_t,i,\varphi \\ \quad (\;force(m^*) = \varphi\mathbf{?}\Box_i/w_t \;\wedge\; w^* = wrld(m^*) \\ \quad \Rightarrow\; (\;W^* = W \cup \{w_t\} \;\wedge\; \longrightarrow^* = \longrightarrow \cup \{(i,w^*,w_t)\}\;)\quad) \end{array}$$

$$\begin{array}{l} frame\_update\_2c((moves,\mathfrak{f}),m^*,\mathfrak{f}^*) \Leftrightarrow \forall i,u,\varphi_1,\varphi_2 \\ \quad (\;(\;force(m^*) \neq \varphi_1\mathbf{?}\Box_i/u \;\wedge\; \neg\exists m' \in moves\;(\\ \qquad m' \overset{g}{\rightarrowtail} m^* \;\wedge\; force(m') = \varphi_2\mathbf{?}\Diamond_i \;\wedge\; defence(m^*)\;)) \\ \quad \Rightarrow\; \mathfrak{f} = \mathfrak{f}^*\quad) \\ \quad \text{where } g = (moves \cup \{m^*\},\mathfrak{f}) \end{array}$$

3. Summary

$$\begin{array}{l} \forall g \forall m^* \forall \mathfrak{f}^* (frame\_update(g,m^*,\mathfrak{f}^*) \Leftrightarrow \\ \quad (\quad frame\_update\_1(g,m^*,\mathfrak{f}^*) \;\wedge\; frame\_update\_2a(g,m^*,\mathfrak{f}^*) \;\wedge \\ \qquad frame\_update\_2b(g,m^*,\mathfrak{f}^*) \;\wedge\; frame\_update\_2c(g,m^*,\mathfrak{f}^*)\quad)\;) \end{array}$$

1.3 Contexts and Context Changes

1. **Frame Binding (F3a)**

$$context\_move\_1((moves,(W^*,\_)),m^*) \;\Leftrightarrow\; wrld(m^*) \in W^*$$

2. **General Context Changes (F3b)**

$$
\begin{aligned}
&\mathit{context\_move\_2a}((\mathit{moves},(W,\longrightarrow^*)),m^*) \;\Leftrightarrow\; \forall w^*,w',i,\varphi\\
&\quad (\;(\;\mathit{defence}(m^*) \;\wedge\; w^* = \mathit{wrld}(m^*) \;\wedge\\
&\qquad \exists m' \in \mathit{moves}\;(\;m' \overset{g}{\rightarrowtail} m^* \;\wedge\; w' = \mathit{wrld}(m') \;\wedge\\
&\qquad\quad (\;\mathit{force}(m') = \varphi\mathbf{?}\Diamond_i \;\vee\; \mathit{force}(m') = \varphi\mathbf{?}\Box_i/w^*\;)\;)\;)\\
&\quad \Rightarrow\; (i,w',w^*) \in\longrightarrow^*\;)\\
&\text{where } g = (\mathit{moves} \cup \{m^*\},(W,\longrightarrow))
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{context\_move\_2b}((\mathit{moves},\mathfrak{f}^*),m^*) \;\Leftrightarrow\; \forall w',u,i,\varphi\\
&\quad (\;(\;\exists m' \in \mathit{moves}\;(\\
&\qquad m' \overset{g}{\rightarrowtail} m^* \;\wedge\; w' = \mathit{wrld}(m') \;\wedge\\
&\qquad \mathit{force}(m') \neq \varphi\mathbf{?}\Diamond_i \;\wedge\; \mathit{force}(m') \neq \varphi\mathbf{?}\Box_i/u\;\;)\;)\\
&\quad \Rightarrow\; \mathit{wrld}(m^*) = w'\;\;)\\
&\text{where } g = (\mathit{moves} \cup \{m^*\},\mathfrak{f}^*))
\end{aligned}
$$

3. **Constructive Backward Change (F3c)**

$$
\begin{aligned}
&\mathit{context\_move\_3con}((\mathit{moves},(W,\longrightarrow)),m^*) \;\Leftrightarrow\\
&\quad (\;(\;\mathit{player}(m^*) = \mathsf{P} \;\wedge\\
&\qquad \mathit{context\_change}(m^*,(\mathit{moves} \cup \{m^*\},(W,\longrightarrow))))\\
&\quad \Rightarrow\; \neg\exists m' \in \mathit{moves}\;(\\
&\qquad \mathit{player}(m') = \mathsf{P} \;\wedge\; \mathit{wrld}(m') = \mathit{wrld}(m^*)\;))
\end{aligned}
$$

4. **Constructive Forward Attack (F3d)**

$$
\begin{aligned}
&\mathit{context\_move\_4con}((\mathit{moves},\_),m^*) \;\Leftrightarrow\\
&\quad (\;(\;\mathit{player}(m^*) = \mathsf{P} \;\wedge\; \mathit{attack}(m^*)\;)\\
&\quad \Rightarrow\; \exists m' \in \mathit{moves}\;(\;\mathit{player}(m') = \mathsf{P} \;\wedge\\
&\qquad \mathit{wrld}(m') = \mathit{wrld}(m^*) \;\wedge\; \mathit{id}(m') < \mathit{id}(n^*)\;)\;)
\end{aligned}
$$

5. **Constructive Box-Diamond-Independence (F3e)**

$$\begin{array}{l} \textit{context\_move\_5con\_a}((\textit{moves}, \_), m^*) \ \Leftrightarrow \ \forall n^*, n', n^\#, u^*, w', \varphi, \psi \\ \quad (\ m^* = (n^*, (\mathsf{P}, \mathbf{!}, \varphi), u^*, n') \ \wedge \\ \quad\quad \exists m' \in \textit{moves}\ (\ m' = (n', (\mathsf{O}, \varphi\mathbf{?}\Diamond_i, \lhd), w', \_) \ \wedge \\ \quad\quad\quad \exists m^\# \in \textit{moves}\ (\ m^\# = (n^\#, (\mathsf{O}, \psi\mathbf{?}\Box_i/u^*, \lhd), w', \_) \ \wedge \\ \quad\quad\quad\quad \neg\exists m^\S \in \textit{moves}\ (\ \textit{wrld}(m^\S) = u^* \ \wedge \ \textit{id}(m^\S) < n^\# \ ) \ ) \ ) \ ) \\ \quad \Rightarrow n' < n^\# < n^* \ ) \end{array}$$

$$\begin{array}{l} \textit{context\_move\_5con\_b}((\textit{moves}, \_), m^*) \ \Leftrightarrow \ \forall n^*, n^\#, u^*, w^*, \rho^*, \varphi, \psi \\ \quad (\ m^* = (n^*, (\mathsf{P}, \varphi\mathbf{?}\Box_i/u^*, \lhd), w^*, \rho^*) \ \wedge \\ \quad\quad \exists m^\# \in \textit{moves}\ (\ m^\# = (n^\#, (\mathsf{O}, \psi\mathbf{?}\Box_i/u^*, \lhd), w^*, \_) \ \wedge \\ \quad\quad\quad \neg\exists m^\S \in \textit{moves}\ (\ \textit{wrld}(m^\S) = u^* \ \wedge \ \textit{id}(m^\S) < n^\# \ ) \ ) \ ) \\ \quad \Rightarrow \rho^* < n^\# < n^* \ ) \end{array}$$

6. Summary

$$\begin{array}{l} \textit{context\_move}(g, m^*) \Leftrightarrow \\ \quad\quad (\quad \textit{context\_move\_1}(g, m^*) \ \wedge \ \textit{context\_move\_2a}(g, m^*) \ \wedge \\ \quad\quad\quad \textit{context\_move\_2b}(g, m^*) \ \wedge \ \textit{context\_move\_3con}(g, m^*) \ \wedge \\ \quad\quad\quad \textit{context\_move\_4con}(g, m^*) \ \wedge \ \textit{context\_move\_5con\_a}(g, m^*) \ \wedge \\ \quad\quad\quad \textit{context\_move\_5con\_b}(g, m^*) \ ) \end{array}$$

2 Soundness Proof

Definition 1. *A Kripke model of CK_n is a structure $\mathcal{M} = (W, \preceq, \longrightarrow, W_\perp, \models)$*

- *W is a non-empty set of worlds.*
- *$\preceq$ is a reflexive and transitive binary relation on W. It is hereditary with respect to propositional variables, that is, for every variable p and worlds w, w', if $w \preceq w'$ and $w \models p$, then $w' \models p$.*
- *$\longrightarrow$ contains binary relations on W.*
- *The subset $W_\perp \subseteq W$ is the set of* fallible worlds *closed under refinement, i.e., $w \in W_\perp$ and $w \preceq w'$ implies $w' \in W_\perp$. $W_\top$ is the set of infallible worlds: $W_\top = W \setminus W_\perp$.*
- *$\models$ is a relation between elements $w \in W$ and propositions A, written $w \models A$,*

or sometimes more explicitly $\mathcal{M}, w \models A$ ("A is satisfied at w in $\mathcal{M}$") with the following properties:

$$
\begin{array}{lll}
w \models \top & & \\
w \models \bot & \textit{iff} & w \in W_\bot \\
w \models \neg A & \textit{iff} & \forall w' \in W_\top (w \preceq w' \Rightarrow w' \not\models A) \\
w \models A \wedge B & \textit{iff} & w \models A \ \ \textit{and} \ \ w \models B \\
w \models A \vee B & \textit{iff} & w \models A \ \ \textit{or} \ \ w \models B \\
w \models A \supset B & \textit{iff} & \forall w' \in W_\top ((w \preceq w' \wedge w' \models A) \Rightarrow w' \models B) \\
w \models \Box_i A & \textit{iff} & \forall w' \in W_\top (w \preceq w' \Rightarrow \forall u \in W (w' \xrightarrow{i} u \Rightarrow u \models A)) \\
w \models \Diamond_i A & \textit{iff} & \forall w' \in W_\top (w \preceq w' \Rightarrow \exists u \in W (w' \xrightarrow{i} u \wedge u \models A))
\end{array}
$$

- *Fallible entities are information-wise maximal elements and therefore all propositional variables $p \in Var$ are valid in them:*

$$w \in W_\bot \Rightarrow w \models p$$

☺

2.1 Constraint System

In the following we describe a *constraint system* we use for the soundness proof. A very similar constraint system for $c\mathcal{ALC}$ is introduced by [Sch15] and used for a correctness proof of a tableau-calculus. The following definitions are adapted from his approach.

For the following constraint system, we consider a different set of worlds $W_c = \mathbb{N} \times \mathbb{N}$. For a $(w, r) \in W_c$ we call w the *world group* and r the *refinement in w*.

Definition 2 (Constraint, Constraint System, *adapted from* [Sch15])**.** *A constraint is a syntactic object of the following form*

$$(w, r) : +\phi, \quad (w, r) : -\phi, \quad (w, r) \preceq (w, r') \quad (w, r) \xrightarrow{i} (u, s), \quad (w, r) : -_{\Box i}\psi$$

where (w, r), (w, r') and (u, s) are worlds of W_c, ϕ and ψ are propositions and $\xrightarrow{i}$ represents a binary relation on W_c.

A constraint system is a pair $S = (\mathcal{C}, \mathcal{A})$ where $\mathcal{C}$ is a finite, non-empty set of constraints *and $\mathcal{A} \subseteq W_c$ is a set of worlds, called the* active set *of S, such that every element of $\mathcal{A}$ occurs in at least one of the constraints of $\mathcal{C}$. The set of variables occuring in $\mathcal{C}$ is called the* support *of S, written $Supp(S)$. Note that $\mathcal{A} \subseteq Supp(S) \subseteq W_c$ and $Supp(S)$ is not empty.*

Sometimes we write $Supp(\mathcal{C})$ instead of $Supp(S)$ as worlds in S appear in $\mathcal{C}$.

Definition 3 (Constraint Satisfiability, *adapted from* [Sch15])**.** *Let $\mathcal{M} = (W, \preceq, \longrightarrow, W_\bot \models)$ be a constructive Kripke-model, $S = (\mathcal{C}, \mathcal{A})$ a constraint system and $\omega : Supp(S) \to W$ a function.*

We say that ω satisfies *a constraint* $c \in \mathcal{C}$ *in* $\mathcal{M}$*, written* $\mathcal{M}, \omega \models_C c$*, according to the following rules:*

$$\begin{array}{lll} \mathcal{M}, \omega \models_C (w,r) : +\phi & \textit{if} & \omega((w,r)) \models \phi \\ \mathcal{M}, \omega \models_C (w,r) : -\phi & \textit{if} & \omega((w,r)) \not\models \phi \\ \mathcal{M}, \omega \models_C (w,r) \xrightarrow{i} (u,s) & \textit{if} & \omega((w,r)) \xrightarrow{i} \omega((u,s)) \\ \mathcal{M}, \omega \models_C (w,r) \preceq (w,r') & \textit{if} & \omega((w,r)) \preceq \omega((w,r')) \\ \mathcal{M}, \omega \models_C (w,r) : -_{\Box i}\psi & \textit{if} & \forall u \in W(\omega(w,r) \xrightarrow{i} u \Rightarrow u \not\models \psi) \end{array}$$

A constraint system $S = (\mathcal{C}, \mathcal{A})$ *is* satisfied *in a model* $\mathcal{M} = (W, \preceq, \longrightarrow, W_\perp, \models)$ *if there exists a function* ω *such that for all* $c \in \mathcal{C}$ *it holds that* $\mathcal{M}, \omega \models_C c$ *and for all worlds* $(w,r) \in \mathcal{A}$ *the assignment* $\omega((w,r))$ *is infallible, i.e.* $\omega((w,r)) \notin W_\perp$*. The pair* $(\mathcal{M}, \omega)$ *is then called* interpretation *of* S*.*

A constraint system S *is called* satisfiable *if it has an interpretation and* unsatisfiable *if it has no interpretation.*

Note that from now on we write $\omega(w,r)$ instead of $\omega((w,r))$.

2.2 Transformation from Moves to Constraints

Now we want to transform plays to constraint systems. For this we need an extra feature $\mathcal{R}$ in the constraint system. It is needed to assign a refinement to a certain move and to retrieve that information later. Because we use two natural numbers for the worlds we have

$$\mathcal{R} \subseteq (\mathbb{N} \times \mathbb{N} \times \mathbb{N}) .$$

The first number refers to a move id, the second one to a world group and the third to a refinement of the world group. For example, if the triple $(3,1,2)$ is in $\mathcal{R}$ then the assertion of the move with the id 3 refers to the world 1 and there to the refinement 2. In fact, one could omit the world group from the tuple as it can be read from the corresponding move. However, we keep it for a better readability.

2.2.1 Transformation Rules

We define a function $cs : \textit{Play} \to \textit{CS}$. It accepts a play and returns the corresponding constraint system. For better reading we write 1 for *init_world*. Note that this function is only defined for valid plays!

$$
\begin{aligned}
& cs(\{m_{\mathbf{H}}\}, \_) = (\mathcal{C}_{\mathbf{H}}, \mathcal{A}_{\mathbf{H}}, \mathcal{R}_{\mathbf{H}}) \\
& \quad \text{where } m_{\mathbf{H}} = (1, (\mathsf{P}, \mathbf{H}, \phi), 1, 0), \\
& \qquad \mathcal{C}_{\mathbf{H}} = \{(1,0) : -\phi\},\ \mathcal{A}_{\mathbf{H}} = \{(1,0)\},\ \mathcal{R}_{\mathbf{H}} = \{(1,1,0)\}
\end{aligned}
$$

$$
\begin{aligned}
& cs(moves, \mathfrak{f}) = cs_{\Delta}(cs(moves_{-1}, \mathfrak{f}), m', m_{last}) \\
& \quad \text{where } |moves| > 1, \quad m_{last} = last(moves), \\
& \qquad moves_{-1} = moves \setminus \{m_{last}\}, \\
& \qquad m' \in moves_{-1} \ \wedge \ m' \overset{(moves,\mathfrak{f})}{\rightarrowtail} m_{last}
\end{aligned}
$$

So we have an initial constraint system that is initialized with the hypothesis. For the succeeding move the function cs_{Δ} is applied. It updates the old constraint system (the previous one) according to the next move in the game.

$$cs_{\Delta} : CS * Move * Move \to CS$$

The result of this update function depends on two input moves we call m and m^* where m^* is a reaction on m.

In general, we have for every particle rule two cases: one for P and one for O. For $\wedge$ and $\vee$ two more. This makes 16 cases.

Note that the case of an implication-attack performed by P is more complex and requires an extra check. That is why we have recursive *calls* in cs_{Δ}.

We have an auxiliary function $max_{\preceq}$ that returns the *maximal refinement* of a given world (w, r) in the support of a given $\mathcal{C}$:

$$
max_{\preceq}(\mathcal{C}, (w,r)) = \begin{cases} (w,r) & \textit{if} \quad \neg\exists r'((w,r) \preceq (w,r') \in \mathcal{C}) \\ & \textit{else} \quad max_{\preceq}(\mathcal{C}, (w,r')) \text{ where } (w,r) \preceq (w,r') \in \mathcal{C} \end{cases}
$$

As we see later, $max_{\preceq}$ always returns a unique world (Lemma 6).

The following tables shows how the constraint system $(\mathcal{C}, \mathcal{A}, \mathcal{R})$ is transformed to $(\mathcal{C}^*, \mathcal{A}^*, \mathcal{R}^*)$ after move m^* is *added* which is a reaction on move m. Values of auxiliary variables are assigned by the arrows $\leftarrow$. For instance, if for $r \leftarrow$ there is written $(n, w, r) \in \mathcal{R}$ then this means that the value of r can retrieved from $\mathcal{R}$ if n and w are known as there is exactly one suitable tuple in $\mathcal{R}$ (what we show later in Lemma 1). The expression $(\_, w, r') \notin \mathcal{R}$ for $r' \leftarrow$ means that the refinement r' is *fresh*, i.e., the world (w, r') does not appear in $\mathcal{R}$ yet. A new refinement is created.

The lines marked with "cond." define additional propositions that hold when the corresponding constraint transformation is applied. The proof that these conditions hold is given later (see Lemma 4).

1. $\neg$
   Note that in this case, a defence is not possible.

m	$n, (\mathsf{P}, \mathbf{!}, \neg\phi), w, \_$	$n, (\mathsf{O}, \mathbf{!}, \neg\phi), w, \_$
m^*	$n^*, (\mathsf{O}, \neg\phi\mathbf{?}\neg, \phi), w, n$	$n^*, (\mathsf{P}, \neg\phi\mathbf{?}\neg, \phi), w, n$
$\mathcal{C}^*$	$\{(w, r') : +\phi, (w, r) \preceq (w, r')\} \cup \mathcal{C}$	$\{(w, r^*) : -\phi\} \cup \mathcal{C}$
$\mathcal{A}^*$	$\{(w, r')\} \cup \mathcal{A}$	$\mathcal{A}$
$\mathcal{R}^*$	$\{(n^*, w, r')\} \cup \mathcal{R}$	$\{(n^*, w, r^*)\} \cup \mathcal{R}$
$r \leftarrow$	$(n, w, r) \in \mathcal{R}$	$(n, w, r) \in \mathcal{R}$
$r' \leftarrow$	$(\_, w, r') \notin \mathcal{R}$	—
$r^* \leftarrow$	—	$max_{\preceq}(\mathcal{C}, (w, r))$
cond.	$(w, r) : -\neg\phi \in \mathcal{C}$ $(w, r) \in \mathcal{A}$	$(w, r) : +\neg\phi \in \mathcal{C}$

m	$n, (\mathsf{P}, \neg\phi\mathbf{?}\neg, \phi), w, \rho$	$n, (\mathsf{O}, \neg\phi\mathbf{?}\neg, \phi), w, \rho$
m^*	$n^*, (\mathsf{O}, \phi\mathbf{?}x, e^*), w, n$	$n^*, (\mathsf{P}, \phi\mathbf{?}x, e^*), w, n$
$\mathcal{C}^*, \mathcal{A}^*, \mathcal{R}^*$	$cs_\Delta((\mathcal{C}, \mathcal{A}, \mathcal{R}), m^!, m^*)$	
$m^! \leftarrow$	$n, (\mathsf{P}, \mathbf{!}, \phi), w, \rho$	$n, (\mathsf{P}, \mathbf{!}, \phi), w, \rho$

2. $\supset$

m	$n, (\mathsf{P}, \mathbf{!}, \phi \supset \psi), w, \_$	$n, (\mathsf{O}, \mathbf{!}, \phi \supset \psi), w, \_$
m^*	$n^*, (\mathsf{O}, (\phi \supset \psi)\mathbf{?}\supset, \phi), w, n$	$n^*, (\mathsf{P}, (\phi \supset \psi)\mathbf{?}\supset, \phi), w, n$
$\mathcal{C}^*$	$\{(w, r') : +\phi, (w, r') : -\psi, (w, r) \preceq (w, r')\} \cup \mathcal{C}$	$\mathcal{C}$
$\mathcal{A}^*$	$\{(w, r')\} \cup \mathcal{A}$	$\mathcal{A}$
$\mathcal{R}^*$	$\{(n^*, w, r')\} \cup \mathcal{R}$	$\{(n^*, w, r^*)\} \cup \mathcal{R}$
$r \leftarrow$	$(n, w, r) \in \mathcal{R}$	$(n, w, r) \in \mathcal{R}$
$r' \leftarrow$	$(\_, w, r') \notin \mathcal{R}$	—
$r^* \leftarrow$	—	$max_{\preceq}(\mathcal{C}, (w, r))$
cond.	$(w, r) : -\phi \supset \psi \in \mathcal{C}$ $(w, r) \in \mathcal{A}$	$(w, r) : +\phi \supset \psi \in \mathcal{C}$

m	$n, (\mathsf{P}, \phi \supset \psi\mathbf{?}\supset, \phi), w, \rho$		$n, (\mathsf{O}, \phi \supset \psi\mathbf{?}\supset, \phi), w, \rho$	
m^*	$n^*, (\mathsf{O}, \phi\mathbf{?}x, e^*), w, n$	$n^*, (\mathsf{O}, \mathbf{!}, \psi), w, n$	$n^*, (\mathsf{P}, \phi\mathbf{?}x, e^*), w, n$	$n^*, (\mathsf{P}, \mathbf{!}, \psi), w, n$
$\mathcal{C}^*, \mathcal{A}^*, \mathcal{R}^*$	$cs_{\Delta}((\mathcal{C} \cup \{(w, r) : -\phi\},$ $\mathcal{A}, \mathcal{R}), m^!, m^*)$	$(\{(w, r) : +\psi\} \cup \mathcal{C}),$ $\mathcal{A}, (\mathcal{R} \cup \{n^*, w, r\})$	$cs_{\Delta}((\mathcal{C}, \mathcal{A}, \mathcal{R}),$ $m^!, m^*)$	$\mathcal{C}, \mathcal{A}, (\mathcal{R} \cup \{n^*, w, r^*\})$
$r \leftarrow$	$(n, w, r) \in \mathcal{R}$		$(n, w, r) \in \mathcal{R}$	
$r^* \leftarrow$	—	—	—	$max_{\preceq}(\mathcal{C}, (w, r))$
$m^! \leftarrow$	$n, (\mathsf{P}, \mathbf{!}, \phi), w, \rho$	—	$n, (\mathsf{O}, \mathbf{!}, \phi), w, \rho$	—

3. $\wedge$

m	$n, (\mathsf{P}, \mathbf{!}, \phi \wedge \psi), w, \_$		$n, (\mathsf{O}, \mathbf{!}, \phi \wedge \psi), w, \_$	
m^*	$n^*, (\mathsf{O}, (\phi \wedge \psi)\mathbf{?}L, \lhd), w, n$	$n^*, (\mathsf{O}, (\phi \wedge \psi)\mathbf{?}R, \lhd), w, n$	$n^*, (\mathsf{P}, (\phi \wedge \psi)\mathbf{?}L, \lhd), w, n$	$n^*, (\mathsf{P}, (\phi \wedge \psi)\mathbf{?}R, \lhd), w, n$
$\mathcal{C}^*$	$\{(w, r) : -\phi\} \cup \mathcal{C}$	$\{(w, r) : -\psi\} \cup \mathcal{C}$	$\{(w, r^*) : +\phi\} \cup \mathcal{C}$	$\{(w, r^*) : +\psi\} \cup \mathcal{C}$
$\mathcal{A}^*$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$
$\mathcal{R}^*$	$\{(n^*, w, r)\} \cup \mathcal{R}$	$\{(n^*, w, r)\} \cup \mathcal{R}$	$\{(n^*, w, r^*)\} \cup \mathcal{R}$	$\{(n^*, w, r^*)\} \cup \mathcal{R}$
$r \leftarrow$	$(n, w, r) \in \mathcal{R}$	$(n, w, r) \in \mathcal{R}$	$(n, w, r) \in \mathcal{R}$	$(n, w, r) \in \mathcal{R}$
$r^* \leftarrow$	—		$max_{\preceq}(\mathcal{C}, (w, r))$	
cond.	$(w, r) : -\phi \wedge \psi \in \mathcal{C}$ $(w, r) \in \mathcal{A}$		$(w, r) : +\phi \wedge \psi \in \mathcal{C}$	

m	$n, (\mathsf{P}, (\phi \wedge \psi)\mathbf{?}L, \lhd), w, \_$	$n, (\mathsf{P}, (\phi \wedge \psi)\mathbf{?}R, \lhd), w, \_$	$n, (\mathsf{O}, (\phi \wedge \psi)\mathbf{?}L, \lhd), w, \_$	$n, (\mathsf{O}, (\phi \wedge \psi)\mathbf{?}R, \lhd), w, \_$
m^*	$n^*, (\mathsf{O}, \mathbf{!}, \phi), w, n$	$n^*, (\mathsf{O}, \mathbf{!}, \psi), w, n$	$n^*, (\mathsf{P}, \mathbf{!}, \phi), w, n$	$n^*, (\mathsf{P}, \mathbf{!}, \psi), w, n$
$\mathcal{C}^*$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$
$\mathcal{A}^*$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$
$\mathcal{R}^*$	$\{(n^*, w, r)\} \cup \mathcal{R}$	$\{(n^*, w, r)\} \cup \mathcal{R}$	$\{(n^*, w, r)\} \cup \mathcal{R}$	$\{(n^*, w, r)\} \cup \mathcal{R}$
$r \leftarrow$	$(n, w, r) \in \mathcal{R}$	$(n, w, r) \in \mathcal{R}$	$(n, w, r) \in \mathcal{R}$	$(n, w, r) \in \mathcal{R}$

4. $\vee$

m	$n, (\mathsf{P}, \mathbf{!}, \phi \vee \psi), w, \_$	$n, (\mathsf{O}, \mathbf{!}, \phi \vee \psi), w, \_$
m^*	$n^*, (\mathsf{O}, (\phi \vee \psi)\mathbf{?}\vee, \lhd), w, n$	$n^*, (\mathsf{P}, (\phi \vee \psi)\mathbf{?}\vee, \lhd), w, n$
$\mathcal{C}^*$	$\mathcal{C}$	$\mathcal{C}$
$\mathcal{A}^*$	$\mathcal{A}$	$\mathcal{A}$
$\mathcal{R}^*$	$\{(n^*, w, r)\} \cup \mathcal{R}$	$\{(n^*, w, r^*)\} \cup \mathcal{R}$
$r \leftarrow$	$(n, w, r) \in \mathcal{R}$	$(n, w, r) \in \mathcal{R}$
$r^* \leftarrow$	—	$max_{\preceq}(\mathcal{C}, (w, r))$
cond.	$(w, r) : -\phi \vee \psi \in \mathcal{C}$ $(w, r) \in \mathcal{A}$	$(w, r) : +\phi \vee \psi \in \mathcal{C}$

m	$n, (\mathsf{P}, (\phi \vee \psi)\mathbf{?}\vee, \lhd), w, \_$		$n, (\mathsf{O}, (\phi \vee \psi)\mathbf{?}\vee, \lhd), w, \_$	
m^*	$n^*, (\mathsf{O}, \mathbf{!}, \phi), w, n$	$n^*, (\mathsf{O}, \mathbf{!}, \psi), w, n$	$n^*, (\mathsf{P}, \mathbf{!}, \phi), w, n$	$n^*, (\mathsf{P}, \mathbf{!}, \psi), w, n$
$\mathcal{C}^*$	$\{(w, r) : -\phi\} \cup \mathcal{C}$	$\{(w, r) : -\psi\} \cup \mathcal{C}$	$\{(w, r^*) : +\phi\} \cup \mathcal{C}$	$\{(w^*, r) : +\psi\} \cup \mathcal{C}$
$\mathcal{A}^*$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$
$\mathcal{R}^*$	$\{(n^*, w, r)\} \cup \mathcal{R}$	$\{(n^*, w, r)\} \cup \mathcal{R}$	$\{(n^*, w, r^*)\} \cup \mathcal{R}$	$\{(n^*, w, r^*)\} \cup \mathcal{R}$
$r \leftarrow$	$(n, w, r) \in \mathcal{R}$		$(n, w, r) \in \mathcal{R}$	
$r^* \leftarrow$	—		$max_{\preceq}(\mathcal{C}, (w, r))$	

5. $\Box_i$

m	$n, (\mathsf{P}, \mathbf{!}, \Box_i\phi), w, \_$	$n, (\mathsf{O}, \mathbf{!}, \Box_i\phi), w, \_$
m^*	$n^*, (\mathsf{O}, \Box_i\phi\mathbf{?}\Box_i/w^{\#}, \lhd), w, n$	$n^*, (\mathsf{P}, \Box_i\phi\mathbf{?}\Box_i/w^{\#}, \lhd), w, n$
$\mathcal{C}^*$	$\{(w,r) \preceq (w,r'), (w,r') \to (w^{\#},0),$ $(w^{\#},0) : -\phi\} \cup \mathcal{C}$	$\{(w^{\#},0) : +\phi\} \cup \mathcal{C}$
$\mathcal{A}^*$	$\{(w,r'), (w^{\#},0)\} \cup \mathcal{A}$	$\mathcal{A}$
$\mathcal{R}^*$	$\{(n^*, w, r')\} \cup \mathcal{R}$	$\{(n^*, w, r^*)\} \cup \mathcal{R}$
$r \leftarrow$	$\{(n, w, r)\} \in \mathcal{R}$	$\{(n, w, r)\} \in \mathcal{R}$
$r' \leftarrow$	$\{(\_, w, r')\} \notin \mathcal{R}$	—
$r^* \leftarrow$	—	$max_{\preceq}(\mathcal{C}, (w,r))$
cond.	$(w,r) : -\Box_i\phi \in \mathcal{C}$ $(\_, w^{\#}, 0) \notin \mathcal{R}$	$(w,r) : +\Box_i\phi \in \mathcal{C}$

m	$n, (\mathsf{P}, \Box_i\phi\mathbf{?}\Box_i/w^{\#}, \lhd), w, \_$	$n, (\mathsf{O}, \Box_i\phi\mathbf{?}\Box_i/w^{\#}, \lhd), w, \_$
m^*	$n^*, (\mathsf{O}, \mathbf{!}, \phi), w^{\#}, n$	$n^*, (\mathsf{P}, \mathbf{!}, \phi), w^{\#}, n$
$\mathcal{C}^*$	$\mathcal{C}^*$	$\mathcal{C}^*$
$\mathcal{A}^*$	$\mathcal{A}^*$	$\mathcal{A}^*$
$\mathcal{R}^*$	$\{(n^*, w^{\#}, 0)\} \cup \mathcal{R}$	$\{(n^*, w^{\#}, 0)\} \cup \mathcal{R}$

6. $\Diamond_i$

m	$n, (\mathsf{P}, \mathbf{!}, \Diamond_i\phi), w, \_$	$n, (\mathsf{O}, \mathbf{!}, \Diamond_i\phi), w, \_$
m^*	$n^*, (\mathsf{O}, (\Diamond_i\phi)\mathbf{?}\Diamond_i, \lhd), w, n$	$n^*, (\mathsf{P}, (\Diamond_i\phi)\mathbf{?}\Diamond_i, \lhd), w, n$
$\mathcal{C}^*$	$\{(w,r) \preceq (w,r'), (w,r') : -_{\Box}\phi\} \cup \mathcal{C}$	$\mathcal{C}$
$\mathcal{A}^*$	$\{(w,r')\} \cup \mathcal{A}$	$\mathcal{A}$
$\mathcal{R}^*$	$\{(n^*, w, r')\} \cup \mathcal{R}$	$\{(n^*, w, r^*)\} \cup \mathcal{R}$
$r \leftarrow$	$\{(n, w, r)\} \in \mathcal{R}$	$\{(n, w, r)\} \in \mathcal{R}$
$r' \leftarrow$	$\{(\_, w, r')\} \notin \mathcal{R}$	—
$r^* \leftarrow$	—	$max_{\preceq}(\mathcal{C}, (w,r))$
cond.	$(w,r) : -\Diamond_i\phi \in \mathcal{C}$ $(w,r) \in \mathcal{A}, (\_, w, r') \notin \mathcal{R}$	$(w,r) : +\Diamond_i\phi \in \mathcal{C}$

m	$n, (\mathsf{P}, (\Diamond_i\phi)\mathbf{?}\Diamond_i, \lhd), w, \_$	$n, (\mathsf{O}, (\Diamond_i\phi)\mathbf{?}\Diamond_i, \lhd), w, \_$
m^*	$n^*, (\mathsf{O}, \mathbf{!}, \phi), w^{\#}, n$	$n^*, (\mathsf{P}, \mathbf{!}, \phi), w^{\#}, n$
$\mathcal{C}^*$	$\{(w^{\#},0) : +\phi, (w,r) \to (w^{\#},0)\} \cup \mathcal{C}^*$	$\{(w^{\#},0) : -\phi\} \cup \mathcal{C}^*$
$\mathcal{A}^*$	$\mathcal{A}^*$	$\{(w^{\#},0)\} \cup \mathcal{A}^*$
$\mathcal{R}^*$	$\{(n^*, w^{\#}, 0)\} \cup \mathcal{R}$	$\{(n^*, w^{\#}, 0)\} \cup \mathcal{R}$
cond.	$(\_, w^{\#}, \_) \notin \mathcal{R}$	$(w,r') \in \mathcal{A}$ $(w,r') : -_{\Box}\phi \in \mathcal{C}^*$

7. Atoms

m	$n, (\mathsf{P}, \mathbf{!}, A), w, \_$	$n, (\mathsf{O}, \mathbf{!}, A), w, \_$
m^*	$n^*, (\mathsf{O}, A\mathbf{?}, \lhd), w, n$	$n^*, (\mathsf{P}, f^*, e^*), w, n$
$\mathcal{C}^*, \mathcal{A}^*, \mathcal{R}^*$	$\mathcal{C}, \mathcal{A}, (\{(n^*, w, r)\} \cup \mathcal{R})$	$cs_\Delta((\mathcal{C}, \mathcal{A}, \mathcal{R}), m^!, m^?)$
$r \leftarrow$	$\{(n, w, r)\} \in \mathcal{R}$	—
$m^? \leftarrow$	—	P's move that is repeated
$m^! \leftarrow$	—	O's move that is attacked again

m	$n, (\mathsf{O}, A\mathbf{?}, \lhd), w, \_$
m^*	$n^*, (\mathsf{P}, \mathbf{!}, \lhd), w, n$
$\mathcal{C}^*$	$\mathcal{C}$
$\mathcal{A}^*$	$\mathcal{A}$
$\mathcal{R}^*$	$\{(n^*, w, r)\} \cup \mathcal{R}$

2.3 Soundness of Dialogues

For better readability the following proofs refer to monomodal CK but are applicable for CK_n as well.

Lemma 1 (Unique Identifiers)**.** *Assume there is a valid play g and a corresponding constraint system $cs(g) = (\mathcal{C}, \mathcal{A}, \mathcal{R})$ and we add a move $m = (n, (p, f, e), w, \rho)$ to g such that the resulting play g' is valid and we obtain the constraint system $cs(g') = (\mathcal{C}', \mathcal{A}', \mathcal{R}')$. Then if $(n, w^*, r^*) \in \mathcal{R}'$ and $(n, w^\#, r^\#) \in \mathcal{R}$ then $w^* = w^\# = w$ and $r^* = r^\#$.*

Proof. Every move in a valid play has a unique identifier as every play is linear and because it is well-formed **(F1a)**. When a move is added to $cs(g)$ with cs_Δ to obtain $cs(g')$ then exactly one entry $(n, w^*, r^\#)$ is added to $\mathcal{R}$. As n is unique for every move, there cannot be another pair $w^\#, r^\#$ combined with n in $\mathcal{R}$. Further, in all of the cases of cs_Δ, w^* is always equal to w. So we conclude that $w = w^* = w^\#$ and $r^* = r^\#$. ☺

Lemma 2 (Constraint Addition)**.** *Assume there is a valid play g and a corresponding constraint system $cs(g) = (\mathcal{C}, \mathcal{A}, \mathcal{R})$ and we add a move $m = (n, (p, f, e), w, \rho)$ to g such that the resulting play g' is valid and we obtain the constraint system $cs(g') = (\mathcal{C}', \mathcal{A}', \mathcal{R}')$.*

1. *If $p = \mathsf{P}$ and m is a reaction on a move m_{O} with $e \neq \lhd$ and m is not an attack on an implication then there is an r such that $(n, w, r) \in \mathcal{R}'$ and $(w, r) : -e \in \mathcal{C}'$.*
2. *If $p = \mathsf{O}$ and m is a reaction on a move m_{P} with $e \neq \lhd$ then there is an r such that $(n, w, r) \in \mathcal{R}'$, $(w, r) : +e \in \mathcal{C}'$ and $(w, r) \in \mathcal{A}'$.*

Proof. For the proof note that constraints are never removed once they have been added to $\mathcal{C}$. Therefore they must be present at all times after the addition.

1. We consider all possible move types for P:

- m is an attack on a negation or a defence on a $\vee$-attack or on a $\Diamond$-attack.
  With cs_Δ, $(w,r) : -e$ is added directly to $\mathcal{C}$ and $\mathcal{R}$ is extended accordingly.
- m is a defence on a $\supset$-attack, on a $\wedge$-attack or on a $\Box$-attack.
  With cs_Δ, $(w,r) : -e$ has been added with O's attack before so that it is already in $\mathcal{C}$. $\mathcal{R}$ has been extended accordingly, as r is the same refinement as the refinement of O's attack.
- m is an attack on an implication.
  This case is excluded.
- For all other cases we have $e = \lhd$, so they are excluded.

2. We consider all possible move types for O:

   - m is an attack on a negation, on an implication or a defence on a $\supset$-attack, on a $\vee$-attack or on a $\Diamond$-attack.
     With cs_Δ, $(w,r') : +e$ is added directly to $\mathcal{C}$ and $\mathcal{R}$ is extended accordingly.
   - m is a defence on a $\wedge$-attack or on a $\Box$-attack.
     With cs_Δ, $(w,r) : +e$ has been added with O's attack before so that it is already in $\mathcal{C}$. $\mathcal{R}$ has been extended accordingly, as r is the same refinement as the refinement of P's attack.
   - For all other cases we have $e = \lhd$, so they are excluded.

☺

Lemma 3 (Active Worlds)**.** *Let g be a valid play with $cs(g) = (\mathcal{C}, \mathcal{A}, \mathcal{R})$. For all moves $m' = (n', (p', f', e'), w', \rho')$: If we add m' to g to obtain a valid g' and the constraint system $cs(g') = (\mathcal{C}', \mathcal{A}', \mathcal{R}')$, then, for the r' from $(n', w', r') \in \mathcal{R}'$ we have $(w', r') \in \mathcal{A}'$.*

Proof by induction on the length of g'. Base case: the first move after the hypothesis is an attack performed by O as it is his turn and there is nothing to defend **(F1b)**. The world 1 that is used in the hypothesis together with its refinement 0 has already been added to the active set (see definition of cs). With O's attack she either refers to the same pair $(1,0)$ which must be in $\mathcal{A}'$ (nothing is ever removed from the active set) or she introduces a new refinement $(1,r')$ that has not existed before. However, in all of these cases, $(1,r')$ is added to the active set by cs_Δ so that it is an element of $\mathcal{A}'$ after the move.

Inductive step: two cases with respect to the player of m'

- $p' = \mathsf{P}$
  With his moves P is never allowed to introduce new worlds **(F2a)**. So P always refers to a world-refinement pair that has been introduced by O before. In general,

every world $(w', r) \in Supp(cs(g))$ that is introduced by O with m' is added directly to the active set with one exception: when O defends against a $\Diamond$-attack. In this case, a new context $(w^{\#}, 0)$ is opened which is not added to $\mathcal{A}$. However, P is not allowed to access it, i.e., state assertions for this world, unless he uses a defence to state an assertion there **(F3d)**. The only possibility for this is that P defends against a $\Diamond$-attack stated by O (**(F3b)**, **(F3e)**). With this defence, $(w^{\#}, 0)$ is added to the active set.
So (w', r) has either been introduced and added to $\mathcal{A}$ with an O-move or the addition to $\mathcal{A}$ has been achieved by P's defence on a $\Diamond$-attack. Therefore we conclude that $(w', r) \in \mathcal{A}$.

- $p' = \mathsf{O}$
If the world (w', r) has already been used, i.e. there is an n such that $(n, w', r) \in \mathcal{R}$, then it must also be in $\mathcal{A}'$ by hypothesis.
Otherwise it is always added directly to the active set with m' in cs_Δ.

☺

Lemma 4 (Invariance)**.** *For all moves m, m' and valid plays g, if m is a move in g and m' can be added to g according to the frame rules resulting in game g', and $m \overset{g'}{\rightarrowtail} m'$ then the conditions for $cs_\Delta(cs(g), m, m')$ hold.*

Proof. By induction on the length of play g':
Base cases: the initial constraint system $cs_1 = (\mathcal{C}_1, \mathcal{A}_1, \mathcal{R}_1)$ with the formula φ stated as hypothesis, consists of $\mathcal{C}_1 = \{(1,0) : -\varphi\}$, $\mathcal{A}_1 = \{(1,0)\}$ and $\mathcal{R}_1 = \{(1,1,0)\}$. Only attacks by O are possible as reactions **(F1b)**. We consider these in detail:

- φ is an atom. No change of cs_1, no conditions. (holds)

- $\varphi = \neg\phi \;\Rightarrow\; m_1 = (1, (\mathsf{P}, \mathbf{H}, \neg\phi), 1, 0)$
$(w, r) \leftarrow (1, 0)$. $(1,0) \in \mathcal{A}_1$ (holds), $(1,0) : -\neg\phi \in \mathcal{C}_1$ (holds).

- $\varphi = \phi \supset \psi \;\Rightarrow\; m_1 = (1, (\mathsf{P}, \mathbf{H}, \phi \supset \psi), 1, 0)$
$(w, r) \leftarrow (1, 0)$, $(w, r') \leftarrow (1, 1)$.
$(1,0) \in \mathcal{A}$ (holds), $(1,0) : -\phi \supset \psi \in \mathcal{C}$ (holds), $\neg\exists n((n, 1, 1) \in \mathcal{R})$ (holds by assignment of r').

- The cases of $\varphi = \phi \wedge \psi$, $\varphi = \phi \vee \psi$ and $\varphi = \Diamond\phi$ are considered analogously.

- $\varphi = \Box\phi \;\; m_1 = (1, (\mathsf{P}, \mathbf{H}, \Box\phi), 1, 0)$
$(w, r) \leftarrow (1, 0)$, $(w, r') \leftarrow (1, 1)$.
O introduces new target entity $w^{\#}$, because he may do so (context strategy, **(F2a)**, **(F2b)**).
$\Rightarrow \;\; \neg\exists n((n, w^{\#}, 0) \in \mathcal{R})$. The other conditions hold as shown above.

Inductive Step: We consider the moves m and m' in a valid play g' such that $m \overset{g'}{\rightarrowtail} m'$.

1. P claims — O attacks — P defends

   a) Negation
   $m = (n, (\mathsf{P}, f, \neg\phi), w, \rho)$ — $m' = (n', (\mathsf{O}, \neg\phi\textbf{?}\neg, \phi), w, n)$
   ((**F1a**), (**F1b**), (**F3b**))
   If m is an attack on an implication, then there is an r such that $(n, w, r) \in \mathcal{R}$ and $(w, r) : -\neg\phi \in \mathcal{C}$ due to cs_Δ. The constraint is also present if m is not an attack on an implication (Lemma 2).
   According to Lemma 3 (w, r) is also in the active set $\mathcal{A}$.

   b) Implication

      i. $m = (n, (\mathsf{P}, f, \phi \supset \psi), w, \rho)$ — $m' = (n', (\mathsf{O}, \phi \supset \psi\textbf{?}\supset, \phi), w, n)$
      ((**F1a**), (**F1b**), (**F3b**))
      To show that the conditions hold we can use the same argumentation as used for the negation.

      ii. $m' = (n', (\mathsf{O}, \phi \supset \psi\textbf{?}\supset, \phi), w, n)$ — $m'' = (n'', (\mathsf{P}, \textbf{!}, \psi), w, n')$
      ((**F1a**), (**F1b**), (**F3b**))
      No conditions, so nothing to show.

   c) Conjunction, Disjunction
   Same argumentation as for implication.

   d) Box

      i. $m = (n, (\mathsf{P}, f, \Box\phi), w, \rho)$ — $m' = (n', (\mathsf{O}, \Box\phi\textbf{?}\Box/w^*, \lhd), w, n)$
      ((**F1a**), (**F1b**), (**F3b**))
      There is an r such that $(n, w, r) \in \mathcal{R}$ and the constraint $(w, r) : -\Box\phi$ is in $\mathcal{C}$ (see argumentation above). As we assume that O introduces new contexts whenever this is possible (context strategy), w^* is a new context. Therefore, there is no $n^\S$ such that $(n^\S, w^\#, 0) \in \mathcal{R}$.

      ii. $m' = (n', (\mathsf{O}, \Box\phi\textbf{?}\Box/w^*, \lhd), w, n)$ — $m'' = (n'', (\mathsf{P}, \textbf{!}, \phi), w^*, n')$
      ((**F1a**), (**F1b**), (**F3b**))
      No conditions, so nothing to show.

   e) Diamond

      i. $m = (n, (\mathsf{P}, f, \Diamond\phi), w, \rho)$ — $m' = (n', (\mathsf{O}, \Diamond\phi\textbf{?}\Diamond, \lhd), w, n)$
      ((**F1a**), (**F1b**), (**F3b**))
      Same argumentation as for negation.

      ii. $m' = (n', (\mathsf{O}, \Diamond\phi\textbf{?}\Diamond, \lhd), w, n)$ — $m'' = (n'', (\mathsf{P}, \textbf{!}, \phi), w^*, n')$
      ((**F1a**), (**F1b**), (**F3b**))
      The world (w, r') has been added to the active set with O's attack, the constraint $(w, r') : -_\Box\phi$ has been added to the constraint system with

that attack, as well.

f) Atom
No conditions, so nothing to show.

2. O claims — P attacks — O defends
For all attack-cases the only condition is that there is an r such that $(n, w, r) \in \mathcal{R}$ and the constraint $(w, r) : +\varphi$ is in $\mathcal{C}$, where φ is the formula stated by O and which is now attacked. We have already shown that this is fulfilled (Lemma 2).

The additional condition that there is no $n^\S$ and no $r^\S$ such that $(n^\S, w^\#, r^\S) \in \mathcal{R}$ when O performs a $\Diamond$-defence is also consequential, as O introduces new worlds whenever this is possible (context strategy, **(F2a)**, **(F2b)**).

☺

Lemma 5 (O's references)**.** *If O performs a move m in a valid play g, she always reacts on the previous move which is performed by P.*
Formally: $\quad m = (n, (O, \_, \_), \_, \rho) \quad \Rightarrow \quad \rho = n - 1.$

Proof. By induction on the length of g.
Base case: O's first move is a reaction on the hypothesis (trivial).
Hypothesis: let $n \geq 2$. $\forall k \in \mathbb{N}$. $1 \leq k \leq n$,
if $m = (k, (\_, \_, \_), \_, \rho)$ and k is odd (O's move) then $\rho = k - 1$.
Induction step: An O-move $m = (n, (\mathsf{O}, f, \varphi), w, \rho)$ is added to g. Every move of P may only be attacked once **(F1d)**. By hypothesis, O always reacted on the last P-move before. Therefore, O already reacted on all moves of P in g with the exception of P's last move. As O may only react on every P-move once, she has to react on the last P-move now which has the id $n - 1$. ☺

Lemma 6 (Linear Refinement Structure)**.** *The refinement structure in a constraint system is always linear:*
Let g be a valid play and $cs(g) = (\mathcal{C}, \mathcal{A}, \mathcal{R})$ the corresponding constraint system. For all worlds $(w, r), (w, r'), (w, r^*)$: If $(w, r) \preceq (w, r') \in \mathcal{C}$ and $(w, r) \preceq (w, r^*) \in \mathcal{C}$ then $r' = r^*$.

Proof. By induction on the length of g:
Base case: only the hypothesis is in g, so only one world $(1, 0)$ in $\mathit{Supp}(cs(g))$. No refinements, so it holds.

Inductive step: We add a move m' to g, so we get g' and $cs(g') = (\mathcal{C}', \mathcal{A}', \mathcal{R}')$. Refinement constraints are only introduced with O's attacks so we only need to consider the corresponding cases. So let us assume that O introduces the new constraint $(w, r) \preceq (w, r')$ with m' in cs_Δ. We then have to show that there is no constraint $(w, r) \preceq (w, r^*)$ for an arbitrary r^* in $\mathcal{C}$. Nothing is ever removed from the constraint system so it is enough to

check this last state.
O always reacts on the last move m performed by P (Lemma 5) that refers to world (w, r). O can't react on an atom-defence by P (no further attack possible) in the followeing this case is excluded.

- m is an attack.
  To obtain the refinement of a P-attack, the maximal refinement of the current world-group is always used ($max_{\preceq}$, cs_{Δ}). By hypothesis, this maximal refinement must be unique. O reacts on this and introduces a new refinement (w, r') of (w, r). By hypothesis, the refinement structure in $\mathcal{C}$ is linear. As (w, r) is maximal in $\mathcal{C}$ there is no $r^* \neq r'$ such that $(w, r) \preceq (w, r^*) \in \mathcal{C}$ and therefore also in $\mathcal{C}'$. So (w, r') is the new maximal refinement.

- m is a defence.
  - m is a defence on **?**L, **?**R, **?**$\vee$ or **?**$\supset$.
    The move is a reaction on O's last attack m_{O} in g **(F1e)**. Only O introduces refinements and this is only possible with attacks (cs_{Δ}). There can't be other O-attacks between m_{O} and m because of the intuitionistic defence restriction so all refinements between m_{O} and m must be the same. Therefore the refinement of m_{O} and of m are maximal. By hypothesis, there is no branching in the refinement-structure so far. With the new refinement (w, r') we have a new maximum, but still no branching.
  - m is a defence on **?**$\Box/w^{\#}$ or **?**$\Diamond$.
    The world $w^{\#}$ has been introduced by O some time before **(F2a)** in w. With that move, $(w^{\#}, 0)$ has been introduced somewhere in $cs(g)$. O always introduces new worlds whenever this is possible (context strategy, **(F2a)**, **(F2b)**). So the only access to $w^{\#}$ is possible from w. P is not allowed to attack formulas in worlds where he has not been before **(F3d)**. Therefore the only possibilities to access $w^{\#}$ is by defence on a $\Box$ or a $\Diamond$ (**(F3d)**, **(F3b)**). Only after one of these possible moves, O can attack P in $w^{\#}$ and the refinements of $(w^{\#}, 0)$ can be constructed (cs_{Δ}). But then P is in $w^{\#}$ and can't go back to w **(F3c)**. So after his defence, P is in $w^{\#}$ for the first time and O has not attacked P there yet, that is why there can't be a refinement of $(w^{\#}, 0)$ in $cs(g)$ and $(w^{\#}, 0)$ is maximal.
    By hypothesis, there is no branching in the refinement-structure so far. With O's new move a new refinement r' of $(w^{\#}, 0)$ is introduced. As $(w^{\#}, 0)$ is maximal in $cs(g)$, $(w^{\#}, r')$ is now maximal in $cs(g')$ and there is no other refinement of $(w^{\#}, 0)$.

☺

Lemma 7 (Monotonic Refinements)**.** *Let g be a valid play with $cs(g) = (\mathcal{C}, \mathcal{A}, \mathcal{R})$.*
(1) If a valid move $m = (n, (p, f, e), w, \rho)$ is added to g so that we obtain g' and the corresponding constraint system $cs(g') = (\mathcal{C}', \mathcal{A}', \mathcal{R}')$ with $(n, w, r') \in \mathcal{R}'$ then there is no

r' such that $(w, r) \preceq (w, r') \in \mathcal{C}'$.
(2) If a constraint of the form $(w, r') : +\varphi$, $(w, r') : -\varphi$ or $(w, r') : -_\Box\varphi$ is added to $\mathcal{C}$ with m then (w, r') is maximal.

Proof. By induction on the length of g'.
Base case: At the beginning, there is only one world with one refinement $(1, 0)$. O's attacks on the hypothesis either refer to the same refinement as the hypothesis-move or create a new refinement which is then maximal (e.g. $(1, 1)$).
Hypothesis: Let g be a valid play with $cs(g) = (\mathcal{C}, \mathcal{A}, \mathcal{R})$. If a valid move $m = (n, (p, f, e), w, \rho)$ is added to g so that we obtain g' and we have the corresponding constraint system $cs(g') = (\mathcal{C}', \mathcal{A}', \mathcal{R}')$ then $(n, w, r') \in \mathcal{R}$ and there is no r'' such that $(w, r') \preceq (w, r'') \in \mathcal{C}'$.
If a constraint of the form $(w, r') : +\varphi$, $(w, r') : -\varphi$ or $(w, r') : -_\Box\varphi$ is added to $\mathcal{C}$ with m then (w, r') is maximal.
Induction step: A new valid move m is added to g.

- Case 1: O attacks.
  In all attacks, O reacts on the last move performed by P (Lemma 5). By hypothesis, this P-move refers to a maximal refinement. O either reuses this refinement (when he attacks $\wedge$) or he creates a new one which is then maximal.

- Case 2: P defends.

  - P defends against a $\supset$-, $\wedge$- or atom-attack.
    The move is a reaction on O's *last* attack m' **(F1e)**. Only O introduces refinements and this is only possible with attacks (cs_Δ). There can't be other O-attacks between m' and m because of the intuitionistic defence restriction so all refinements between m' and m must be the same. Therefore the refinement of m must be equal to the refinement of O's last move which is maximal by hypothesis.

  - P defends against a $\vee$-attack.
    The maximal refinement is added according to $cs(g)$. This refinement is unique according to Lemma 6.

  - P defends against a $\Box$- or $\Diamond$-attack. The world $w^\#$ has been introduced by O some time before **(F2a)** in w. With that move, $(w^\#, 0)$ has been introduced somewhere in $cs(g)$. Note that O always introduces new worlds whenever this is possible (context strategy, **(F2a)**, **(F2b)**). So the only access to $w^\#$ is possible from w. P is not allowed to attack formulas in worlds where he has not been before **(F3d)**. So the only possibilities to access $w^\#$ is by a defence on a $\Box$ or on a $\Diamond$ (**(F3d)**, **(F3b)**). Only after one of these possible moves, O can attack P in $w^\#$ and then refinements of $(w^\#, 0)$ can be constructed (cs_Δ). But then P is in $w^\#$ and can't go back to w **(F3c)**. So after his defence, P is in $w^{\#}$ for the first time and O has not attacked P there yet so there can't be a refinement of $(w^{\#}, 0)$ in $cs(g')$, therefore $(w^{\#}, 0)$ is maximal.

- Case 3: P attacks.
  For the attacking moves, the refinements are always set to the maximum (cs_{Δ}). According to Lemma 6 these refinements are always unique.

- Case 4: O defends.

  – O defends against a $\supset$-, $\wedge$- or $\vee$-attack.
    O always reacts on the last move performed by P (Lemma 5). With O's defence, the refinement of that move (which is an attack) is taken over (cs_{Δ}). By hypothesis, this refinement is maximal. Therefore, the refinement of the defence must be maximal as well.

  – O defends against a $\Box$- or $\Diamond$-attack.
    When O accesses $w^{\#}$ from w, we have to consider two cases:

    1. O has not been in $w^{\#}$ before.
       Then she hasn't been able to construct refinements there yet. That's why $(w^{\#}, 0)$ is maximal.

    2. O has been in $w^{\#}$ but returned back to w with a reaction on one of P's moves.
       P must still be in w as O always reacts on P's last move (Lemma 5). However P can never go back to a previously visited context **(F3c)**. So he mustn't have been in $w^{\#}$ yet. Therefore O has not performed any attacks in $w^{\#}$ and that's why there can't be any refinements of $(w^{\#}, 0)$ in $cs(g')$. This means that $(w^{\#}, 0)$ is maximal.

    There are no other possibilities for O to return to a previously visited context.

  – O defends against an atom-attack.
    As P may not change back to previously visited contexts **(F3c)** he is also not able to repeat attacks for previously visited contexts. With P's attack, the refinement of that attack is maximized (case 3) and O defends against this attack with a maximal refinement (the reasons stated above hold here as well).

☺

Theorem 1 (Unsatisfiability of Plays)**.** *Let g be a valid play.*

1. *If O is the player of the last move of g and there exists a play g' such that $(g, g') \in$ FRuleSet for which the constraint system $cs(g')$ is not satisfiable then $cs(g)$ is also not satisfiable.*

2. *If P is the player of the last move of g and the constraint systems $cs(g')$ of all plays*

g' such that $(g, g') \in FRuleSet$ are not satisfiable then $cs(g)$ is also not satisfiable.

Proof. 1. We consider the valid play g with its constraint system $cs(g) = (\mathcal{C}, \mathcal{A}, \mathcal{R})$. We have to show for every possible sort of move performed by P separately that for one of the corresponding moves, if the constraint system $cs(g')$ we have after the move is unsatisfiable then $cs(g)$ is also unsatisfiable.

Note that in the following f is considered to be **!**, otherwise it must be either $\varphi\mathbf{?}\neg$ or $\varphi\mathbf{?}\supset$ for some φ. However these cases are also handled as defences in cs_Δ.

?$\neg$ a) $m = (n, (\mathsf{O}, f, \neg\phi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{P}, \neg\phi\mathbf{?}\neg, \phi), w, n)$
Suppose g' is the result of P's attack on a negation in g performed by O and $c' \in \mathcal{C}'$ such that $c' = (w, r) : -\phi$ (by invariance).
Let us assume that $cs(g)$ is satisfiable with $(w, r) : +\neg\phi \in \mathcal{C}$ (by invariance), i.e., there is an interpretation $\mathcal{M}, \omega$ such that $\mathcal{M}, \omega \models_C +\neg\phi$, so $\omega(w, r) \models \neg\phi$. This means that for every non-fallible refinement w^* of $\omega(w, r)$ it holds that $w^* \not\models \phi$. Particularly, $\omega(w, r) \not\models \phi$ as every world is a refinement of itself (reflexivity). This can be expressed by the constraint c'. As $\mathcal{C}' = \mathcal{C} \cup \{c'\}$, $cs(g')$ must be satisfiable.

b) $m = (n, (\mathsf{O}, \neg\phi\mathbf{?}\neg, \phi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{P}, \phi\mathbf{?}x, e'), w, n)$
This is a counter-attack which is handled as an attack on a defence $(n, (\mathsf{P}, \mathbf{!}, \phi), w, \rho)$ in cs_Δ. The constraint system is affected accordingly (see corresponding case in this proof).

?$\supset$, **!**$\supset$ a) $m = (n, (\mathsf{O}, f, \phi \supset \psi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{P}, \phi \supset \psi\mathbf{?}\supset, \phi), w, n)$
The constraint system does not change. So if $cs(g)$ is satisfiable then $cs(g')$ must also be satisfiable.

b) $m = (n, (\mathsf{O}, \phi \supset \psi\mathbf{?}\supset, \phi), w, \rho)$
$\rightarrowtail \quad m'_1 = (n', (\mathsf{P}, \phi\mathbf{?}x, e'_1), w, n) \quad$ or $\quad m'_2 = (n', (\mathsf{P}, \mathbf{!}, \psi), w, n)$
Here, we discuss both possibilities of P's reaction on O's attack on an implication: P can defend against this attack by stating ψ which does not modify $\mathcal{C}$ or he may perform a counter-attack, what means that the constraint system can be changed by another rule. In the latter case, cs_Δ treats m as a defence and the constraint system is modified accordingly (see corresponding case in proof). The current rule does not touch it further.
In the former case $\mathcal{C}'$ is the same as $\mathcal{C}$ as nothing is changed.

?$\wedge$ $m = (n, (\mathsf{O}, f, \phi \wedge \psi), w, \rho)$
$\rightarrowtail \quad m'_1 = (n', (\mathsf{P}, \phi \wedge \psi\mathbf{?}L, \lhd), w, n) \quad$ or $\quad m'_2 = (n', (\mathsf{P}, \phi \wedge \psi\mathbf{?}R, \lhd), w, n)$
Suppose g' is the result of P's attack on a conjunction in g performed by O. So there must be a constraint $c'_1 \in \mathcal{C}'$ or $c'_2 \in \mathcal{C}'$ such that $c'_1 = (w, r) : +\phi$ and $c'_2 = (w, r) : +\psi$.

Let us assume that $cs(g)$ is satisfiable with $c = (w,r) : +\phi \wedge \psi \in \mathcal{C}$ (invariance). There is an interpretation $\mathcal{M}, \omega$ such that $\omega(w,r) \models \phi$ and $\omega(w,r) \models \psi$. This can be expressed by the constraint c'_i. As $\mathcal{C}' = \mathcal{C} \cup \{c'_i\}$, $cs(g')$ must be satisfiable.

!$\wedge$ $m_1 = (n, (\mathsf{O}, \phi \wedge \psi \textbf{?}L, \lhd), w, \rho) \quad \rightarrowtail \quad m'_1 = (n', (\mathsf{P}, \textbf{!}, \phi), w, n)$
$m_2 = (n, (\mathsf{O}, \phi \wedge \psi \textbf{?}R, \lhd), w, \rho) \quad \rightarrowtail \quad m'_2 = (n', (\mathsf{P}, \textbf{!}, \psi), w, n)$
Suppose g' is the result of P's defence on an attack m_1 or m_2 on a conjunction in g. According to the cs_Δ the constraint system does not change, therefore if $cs(g)$ is satisfiable then $cs(g')$ is also satisfiable.

?$\vee$ $m = (n, (\mathsf{O}, f, \phi \vee \psi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{P}, \phi \vee \psi \textbf{?}\vee, \lhd), w, \rho)$
Suppose g' is the result of P's attack on a disjunction in g. According to cs_Δ the constraint system does not change, therefore if $cs(g)$ is satisfiable then $cs(g')$ is also satisfiable.

!$\vee$ $m = (n, (\mathsf{O}, \phi \vee \psi \textbf{?}\vee, \lhd), w, \rho)$
$\rightarrowtail \quad m'_1 = (n', (\mathsf{P}, \textbf{!}, \phi), w, n) \quad$ or $\quad m'_2 = (n', (\mathsf{P}, \textbf{!}, \psi), w, n)$
We have two possible resulting constraint systems $cs(g'_1) = (\mathcal{C}'_1, \mathcal{A}')$ and $cs(g'_2) = (\mathcal{C}'_2, \mathcal{A}')$ depending on the move which is done (m'_1 or m'_2).
Let's assume that $cs(g)$ is satisfiable. Then $\mathcal{C}$ contains $(w,r) : -\phi \vee \psi$ (invariance). So we have an interpretation $\mathcal{M}, \omega$ with $\omega(w,r) \not\models \phi \vee \psi$ in $\mathcal{M}$. This means that $\omega(w,r) \not\models \phi$ and $\omega(w,r) \not\models \psi$. So we can add the constraint $(w,r) : -\phi$ and the constraint $(w,r) : -\psi$ to $\mathcal{C}$ without changing the satisfiability of $cs(g)$. We then have $cs(g'_1)$ or $cs(g'_2)$ which are therefore both satisfiable.

?$\Box$ $m = (n, (\mathsf{O}, f, \Box\phi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{P}, \Box\phi\textbf{?}\Box/w^\#, \lhd), w, n)$
After the move we have $\mathcal{C}' = \mathcal{C} \cup \{(w^\#, 0) : +\phi\}$. We assume that $cs(g)$ is satisfiable so by the conditions of cs_Δ there is an interpretation $\mathcal{M}, \omega$ such that $\mathcal{M}, \omega \models_C (w,r) : +\Box\phi$ and $\omega(w,r) \models \Box\phi$. We also know that $(w,r) \in \mathcal{A}$, so $\omega(w,r)$ is infallible. This means that for all $u^* \in W$ such that $\omega(w,r) \longrightarrow u^*$ implies $u^* \models +\phi$.

P may not introduce new worlds **(F2a)** and there must be a transition in the play's frame from w (the source world) to $w^\#$ **(F3b)**. Therefore O must have introduced $w^\#$ and the link $w \longrightarrow w^\#$ somewhere before with a defence on a $\Diamond$-attack or a $\Box$-attack **(F2b)**. We call these moves $m_1^\S = (n^\S, (\mathsf{O}, \textbf{!}, \varphi), w^\#, \rho^\S)$ and $m_2^\S = (n^\S, (\mathsf{O}, \chi\textbf{?}\Box/w^\#, \lhd), w^\#, \rho^\S)$.
We obtain the refinement $r^\S$ of w for the move $m_i^\S$ from $\mathcal{R}$ ($(n^\S, w, r^\S) \in \mathcal{R}$, this must be possible because for every new move, such a unique entry is added to $\mathcal{R}$). As $n^\S < n$ we conclude that $\omega(w, r^\S) \preceq \omega(w, r')$ (Lemma 7) where r' is the refinement of assigned to m'.

Case 1: There is a move $m_1^\S = (n^\S, (\mathsf{O}, \textbf{!}, \varphi), w^\#, \rho^\S)$ before m which is a

defence on a $\diamond$-attack. As there must be such an attack, the constraint $(w, r^{\S}) : +\diamond\varphi$ must also be in $\mathcal{C}$ (invariance). By Definition 1 we know that there is a transition from all non-fallible refinements of $\omega(w, r^{\S})$ to $\omega(w^{\#}, 0)$ introduced by O with his move $m^{\S}$. Particularly, we have $\omega(w, r') \longrightarrow \omega(w^{\#}, 0)$, so we can add the constraint $(w^{\#}, 0) : +\phi$ to $\mathcal{C}$ without changing satisfiability.
Case 2: There is a move $m_2^{\S} = (n^{\S}, (\mathsf{O}, \chi\mathbf{?}\Box/w^{\#}, \lhd), w^{\#}, \rho^{\S})$. This must have been stated after m (where $m \overset{g'}{\rightarrowtail} m'$) **(F3e)**, i.e., $\rho < n^{\S} < n'$.
By Lemma 7 we know that $\omega(w, r) \preceq \omega(w, r^{\S}) \preceq \omega(w, r')$ where r is the refinement of move m, i.e. $(\rho, w, r) \in \mathcal{R}$. As $\omega(w, r) \models \Box\phi$ and because $(w, r^{\S}) \to (w^{\#}, 0)$ and $(w, r^{\S}) \in \mathcal{A}$ (added with move $m^{\S}$) we conclude by Definition 1 $\omega(w, r^{\S}) \longrightarrow \omega(w^{\#}, 0) \Rightarrow \omega(w^{\#}, 0) \models \phi$. The condition is true as $(w, r^{\S}) \to (w^{\#}, 0)$ has been added to $\mathcal{C}$ with $m^{\S}$. Therefore $\omega(w^{\#}, 0) \models \phi$ and the constraint $(w^{\#}, 0) : +\phi$ can be added to $\mathcal{C}$ without changing the satisfiability of $cs(g)$.

!$\Box$ $m = (n, (\mathsf{O}, \Box\phi\mathbf{?}\Box/w^{\#}, \lhd), w, \rho) \quad \rightarrowtail \quad m' = (n, (\mathsf{P}, \mathbf{!}, \phi), w^{\#}, n)$
No changes of constraint system, so it holds.

?$\diamond$ $m = (n, (\mathsf{O}, f, \diamond\phi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{P}, \diamond\phi\mathbf{?}\diamond, \lhd), w, n)$
No changes of constraint system, so it holds.

!$\diamond$ $m = (n, (\mathsf{O}, \diamond\phi\mathbf{?}\diamond, \lhd), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{P}, \mathbf{!}, \phi), w^{\#}, n)$
Let us assume that $cs(g)$ is satisfiable. According to Lemma 4 we have the constraint $(w, r') : -_{\Box}\phi$ in $\mathcal{C}$. As $cs(g)$ is satisfiable, there is an interpretation $\mathcal{M}, \omega$ such that $\mathcal{M}, \omega \models_C (w, r') : -_{\Box}\phi$. So for all $u^* \in W$ such that $\omega(w, r') \longrightarrow u^*$ it holds that $u^* \not\models \phi$.

P may not introduce new worlds **(F2a)** and there must be a transition in the play's frame from w (the source world) to $w^{\#}$ **(F3b)**. Therefore O must have introduced $w^{\#}$ and the link $w \longrightarrow w^{\#}$ somewhere before with a defence on a $\diamond$-attack or a $\Box$-attack **(F2b)**. We call these moves $m_1^{\S} = (n^{\S}, (\mathsf{O}, \mathbf{!}, \varphi), w^{\#}, \rho^{\S})$ and $m_2^{\S} = (n^{\S}, (\mathsf{O}, \mathbf{?}\Box/w^{\#}, \lhd), w^{\#}, \rho^{\S})$.
We obtain the refinement $r^{\S}$ of w for the move $m_i^{\S}$ from $\mathcal{R}$ ($(n^{\S}, w, r^{\S}) \in \mathcal{R}$, this must be possible because for every new move, such a unique entry is added to $\mathcal{R}$). As $n^{\S} < n$ we conclude that $\omega(w, r^{\S}) \preceq \omega(w, r)$ (Lemma 7).

Here we can argue the same way we have done for the case **?**$\Box$ (see above). We conclude that $\omega(w, r) \preceq \omega(w^{\#}, 0)$.

Let $\omega(w^{\#}, 0)$ be one of the u^*s. Then we know that $\omega(w^{\#}, 0) \not\models \phi$ because of $(w, r) : -_{\Box}\phi$. We can add $(w^{\#}, 0) : -\phi$ to $cs(g)$ without changing its satisfiability. As $\omega(w^{\#}, 0) \not\models \phi$, $\omega(w^{\#}, 0)$ is infallible and $(w^{\#}, 0)$ can be added to $\mathcal{A}$. We reach $cs(g')$ which must also be satisfiable.

a**?** $m = (n, (\mathsf{O}, f, a), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{P}, f', \phi'), w', n)$

This is a repetition of an attack which has already been done by P in the same world (**(F1c)**, **(F3c)**). The constraint system is affected accordingly (see corresponding case in the prove). There are no other changes.

a**!** $m = (n, (\mathsf{O}, a\mathbf{?}, \lhd), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{P}, \mathbf{!}, \lhd), w, n)$
The constraint system is not changed, so it holds.

2. Let $cs(g) = (\mathcal{C}, \mathcal{A}, \mathcal{R})$ be the constraint system of a valid play g. We have to show for every possible move m' performed by O such that after adding m' to g we obtain the valid play g': if the constraint system $cs(g')$ is unsatisfiable then g is also unsatisfiable.
The single proofs work by contradiction, so we show that if $cs(g)$ is satisfiable then at least one of O's possible moves leads to a satisfiable constraint system as well. We consider all possible moves m of P in g to discuss O's possible reactions m'.

In the following f is again considered to be **!**, otherwise it must be either $\varphi\mathbf{?}\neg$ or $\varphi\mathbf{?}\supset$ for some φ. However these cases are also handled as defences in cs_Δ.

?$\neg$ a) $m = (n, (\mathsf{P}, f, \neg\phi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{O}, \neg\phi\mathbf{?}\neg, \phi), w, n)$
Let us assume that $cs(g)$ is satisfiable. $\mathcal{C}$ contains $(w, r) : -\neg\phi$ by Lemma 4. Therefore there is an interpretation $\mathcal{M}, \omega$ such that $\mathcal{M}, \omega \models_C (w, r) : -\neg\phi$, so $\omega(w, r) \not\models \neg\phi$. By Definition 1 this means that there is a non-fallible world v in $\mathcal{M}$ such that $\omega(w, r) \preceq v$ and that $v \models \phi$. So we can extend ω by introducing (w, r') to our constraint system and defining $\omega(w, r') = v$. Then we add $(w, r) \preceq (w, r')$ and $(w, r') : +\phi$ to $\mathcal{C}$ and (w, r') to $\mathcal{A}$ (as $v \in W_\top$). This does not change satisfiability of $cs(g)$. We then have $cs(g')$ which must then be satisfiable as well.

b) $m = (n, (\mathsf{P}, \neg\phi\mathbf{?}\neg, \phi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{O}, \phi\mathbf{?}x, e'), w, n)$
This is a counter-attack which is handled as an attack on a defence $(n, (\mathsf{P}, \mathbf{!}, \phi), w, \rho)$ in cs_Δ. The constraint system is affected accordingly (see corresponding case in this proof).

?$\supset$, **!**$\supset$ a) $m = (n, (\mathsf{P}, f, \phi \supset \psi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{O}, \phi \supset \psi\mathbf{?}\supset, \phi), w, n)$
We assume that $cs(g)$ is satisfiable. $\mathcal{C}$ contains $(w, r) : -\phi \supset \psi$ by Lemma 4. Therefore there is an interpretation $\mathcal{M}, \omega$ such that $\omega(w, r) \not\models \phi \supset \psi$ in $\mathcal{M}$. By Definition 1 this means that there is a non-fallible world v in $\mathcal{M}$ such that $\omega(w, r) \preceq v$ and that $v \models \phi$ and $v \not\models \psi$.
So if we extend $cs(g)$ by a new world (w, r') such that $\omega(w, r') = v$. Then we add $(w, r) \preceq (w, r')$, $(w, r') : +\phi$ and $(w, r') : -\psi$ to $\mathcal{C}$ and (w, r') to $\mathcal{A}$ (as $v \in W_\top$). This does not change satisfiability of $cs(g)$. We then have $cs(g')$ which must then be satisfiable as well.

b) $m = (n, (\mathsf{P}, \phi \supset \psi\mathbf{?}\supset, \phi), w, \rho)$
$\rightarrowtail \quad m'_1 = (n', (\mathsf{O}, \phi\mathbf{?}x, e'_1), w, n) \quad$ or $\quad m'_2 = (n', (\mathsf{O}, \mathbf{!}, \psi), w, n)$
Here, we discuss both possibilities of O's reaction on P's attack on an

implication: O can defend against this attack by stating ψ which causes that $c'_2 = (w,r) : +\psi$ is added to the constraints (we call the corresponding game g'_2) or she may perform a counter-attack, what means that $c'_1 = (w,r) : -\phi$ is added to the constraint system plus maybe some extra constraints according to the kind of attack (g'_1). For the latter case cs_Δ continues treating m as a defence and m'_1 as an attack on it. The constraint system is affected accordingly (see corresponding case in this proof).

Let us assume that $cs(g)$ is satisfiable, i.e., there is an interpretation $\mathcal{M},\omega$ such that $\mathcal{M},\omega \models_C (w,r) : +\phi \supset \psi$, so $\omega(w,r) \models \phi \supset \psi$.
Now let us also assume that $\omega(w,r) \models \phi$. Then we derive $\omega(w,r) \models \psi$ by the implication. So the constraint $(w,r) : +\psi$ can be added without changing the satisfiability. In this case $cs(g'_2)$ is satisfiable.

If $\omega(w,r) \models \phi$ is not true, we have $\omega(w,r) \not\models \phi$. This can be expressed with the additional constraint $(w,r) : -\phi$. Therefore $cs(g'_1)$ is satisfiable.

?$\wedge$ $m = (n, (\mathsf{P}, f, \phi \wedge \psi), w, \rho)$
$\rightarrowtail \quad m'_1 = (n', (\mathsf{O}, \phi \wedge \psi\mathbf{?}L, \lhd), w, n) \quad$ or $\quad m'_2 = (n', (\mathsf{O}, \phi \wedge \psi\mathbf{?}R, \lhd), w, n)$
We have two possible resulting constraint systems $cs(g'_1) = (\mathcal{C}'_1, \mathcal{A}', \mathcal{R}')$ and $cs(g'_2) = (\mathcal{C}'_2, \mathcal{A}', \mathcal{R}')$ depending on the move which is done (m'_1 or m'_2). Let's assume that $cs(g)$ is satisfiable. Then $\mathcal{C}$ contains $(w,r) : -\phi \wedge \psi$ (invariance). So we have an interpretation $\mathcal{M},\omega$ with $\omega(w,r) \not\models \phi \wedge \psi$ in $\mathcal{M}$. By Definition 1 this means that $\omega(w,r) \not\models \phi$ or $\omega(w,r) \not\models \psi$. So we can either add the constraint $(w,r) : -\phi$ or the constraint $(w,r) : -\psi$ to $\mathcal{C}$. We then have $cs(g'_1)$ or $cs(g'_2)$. At least one of the results must be satisfiable.

!$\wedge$ $m_1 = (n, (\mathsf{P}, \phi \wedge \psi\mathbf{?}L, \lhd), w, \rho) \quad \rightarrowtail \quad m'_1 = (n', (\mathsf{O}, \mathbf{!}, \phi), w, n)$
$m_2 = (n, (\mathsf{P}, \phi \wedge \psi\mathbf{?}R, \lhd), w, \rho) \quad \rightarrowtail \quad m'_2 = (n', (\mathsf{O}, \mathbf{!}, \psi), w, n)$
No change of the constraint system, so it holds.

?$\vee$ $m = (n, (\mathsf{P}, f, \phi \vee \psi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{O}, \phi \vee \psi\mathbf{?}\vee, \lhd), w, \rho)$
No change of the constraint system, so it holds.

!$\vee$ $m = (n, (\mathsf{P}, \phi \vee \psi\mathbf{?}\vee, \lhd), w, \rho)$
$\rightarrowtail \quad m'_1 = (n', (\mathsf{O}, \mathbf{!}, \phi), w, n) \quad$ or $\quad m'_2 = (n', (\mathsf{O}, \mathbf{!}, \psi), w, n)$
Let us assume that $cs(g)$ is satisfiable, i.e., there is an interpretation $\mathcal{M},\omega$ such that $\mathcal{M},\omega \models_C (w,r) : +\phi \vee \psi$, so $\omega(w,r) \models \phi \vee \psi$. So either $\omega(w,r) \models \phi$ or $\omega(w,r) \models \psi$.
Let us assume that $\omega(w,r) \models \phi$, then we can add the constraint $(w,r) : +\phi$ to $\mathcal{C}$ without changing satisfiability. In this case $cs(g')$ is satisfiable.
If $\omega(w,r) \not\models \phi$ then $\omega(w,r) \models \psi$ must hold and we can add $(w,r) : +\psi$ to $\mathcal{C}$. In this case $cs(g')$ is satisfiable, too.

?$\Box$ $m = (n, (\mathsf{P}, f, \Box\phi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{O}, \Box\phi\mathbf{?}\Box/w^{\#}, \lhd), w, n)$

Again, we assume satisfiability of $cs(g)$. By invariance we have $(w, r) : -\Box\phi \in \mathcal{C}$. So there is an interpretation $\mathcal{M}, \omega$ with $\omega(w, r) \not\models \Box\phi$ in $\mathcal{M}$. By Definition 1 this means that there must be a refinement v of $\omega(w, r)$ in $\mathcal{M}$ that is non-fallible and there must also be a world u^* such that $v \longrightarrow u$ and $u \not\models \phi$. We introduce the worlds (w, r') and $(w^\#, 0)$ (which are not in $cs(g)$ yet) and extend ω in such a way that $\omega(w, r') = v$ and $\omega(w^\#, 0) = u^*$. Then we add the constraints $(w, r) \preceq (w, r')$, $(w, r') \to (w^\#, 0)$ and $(w^\#, 0) : -\phi$ to $cs(g)$ which does not change the satisfiability. We reach $cs(g')$ which must also be satisfiable.

!$\Box$ $m = (n, (\mathsf{P}, \Box\phi\textbf{?}\Box/w^\#, \lhd), w, \rho) \quad \rightarrowtail \quad m' = (n, (\mathsf{O}, \textbf{!}, \phi), w^\#, n)$
No changes of constraint system, so it holds.

?$\Diamond$ $m = (n, (\mathsf{P}, f, \Diamond\phi), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{O}, \Diamond\phi\textbf{?}\Diamond, \lhd), w, n)$
We assume that $cs(g)$ is satisfiable. By invariance we have $(w, r) : -\Diamond\phi$ in $\mathcal{C}$. As $cs(g)$ is satisfiable, there is an interpretation $\mathcal{M}, \omega$ such that $\omega(w, r) \not\models \Diamond\phi$. By Definition 1 there must be a non-fallible refinement v of $\omega(w, r)$ in $\mathcal{M}$. Further, for all successors u of v we have $u^* \not\models A$. So we introduce the new world (w, r') in $cs(g)$ and update ω such that $\omega(w, r') = v$. Then it doesn't change satisfiability of $cs(g)$ when we add $(w, r) \preceq (w, r')$ and $(w, r') : -_\Box\phi$ to $\mathcal{C}$ and (w, r') to $\mathcal{A}$ as $v \in W_\top$. We reach $cs(g')$ which must be satisfiable as well.

!$\Diamond$ $m = (n, (\mathsf{P}, \Diamond\phi\textbf{?}\Diamond, \lhd), w, \rho) \quad \rightarrowtail \quad m' = (n, (\mathsf{O}, \textbf{!}, \phi), w^\#, n)$
We have two new constraints $c'_1 = (w^\#, 0) : +\phi$ and $c'_2 = (w, r) \to (w^\#, 0)$ such that $\mathcal{C}' = \mathcal{C} \cup \{c'_1, c'_2\}$.
Let us assume that $cs(g)$ is satisfiable. By the conditions of cs_Δ we have $(w, r) : +\Diamond\phi \in \mathcal{C}$. As $(w, r) \in \mathcal{A}$ and $cs(g)$ is satisfiable we conclude that there is an interpretation $\mathcal{M}, \omega$ such that $\omega(w, r) \models \Diamond\phi$ and $\omega(w, r) \in W_\top$. Therefore, there is a world $u^* \in W$ such that $\omega(w, r) \longrightarrow u^*$ and $u^* \models \phi$.
By assumption, all of O's reactions on g cause unsatisfiability, no matter which world $w^\#$ she chooses in her defence and for which world she states ϕ. So let's claim that $w^\# = u$. Then we can introduce a new world $(w^\#, 0)$ into the constraint system $cs(g)$ such that $u^* = \omega(w^\#, 0)$ and add the constraints $(w, r) \to (w^\#, 0)$ and $(w^\#, 0) : +\phi$ and the resulting constraint system (which corresponds to $cs(g')$) is still satisfiable.

a**?** $m = (n, (\mathsf{P}, f, a), w, \rho) \quad \rightarrowtail \quad m' = (n', (\mathsf{O}, a\textbf{?}, \lhd), w, n)$
No changes of the constraint system, so it holds.

!a refers to another case.

☺

Theorem 2 (Unsatisfiability of Finished Plays)**.** *For all valid finished plays g (finished(g)), if P is the winner of g (winner_of_play(g, P)) then $cs(g)$ is not satisfiable, i.e., there ex-*

ists no interpretation for cs(g).

Proof. P is the winner of the play g. This means there is no possible move for O, in g and P is the last player (see *winning rule*).
Let us assume that $cs(g)$ is satisfiable. By Theorem 1 we know that if a play g, where P is the last player who performed a move, is satisfiable then there must be at least one g' that can be obtained by an O-move added to g and which is also satisfiable. But as P is the winner of g, O can't continue g with any move and therefore there is no g'.
So $cs(g)$ cannot be satisfiable. ☺

Theorem 3 (Unsatisfiability of Dialogues)**.** *For all valid plays g, if P is the winner of the dialogue starting in g (winner_of_dialogue(P, g)) then $cs(g)$ is not satisfiable, i.e., there exists no interpretation for g.*

Proof. By induction on the length of g.
Base case: There is no g' such that $(g, g') \in \mathit{FRuleSet}$. As P is the last player of g, it is O's turn now, but O is not able to move. The game is finished. By Theorem 2 $cs(g)$ is not satisfiable.

Inductive step: we have to distinguish two cases:

1. The last player of g is O. Then there must be a g' such that $(g, g') \in \mathit{FRuleSet}$ and P is the winner of g'.
   By hypothesis, $cs(g')$ is not satisfiable. Therefore, $cs(g)$ is also not satisfiable by Theorem 1 (1).

2. The last player of g is P. Then for all g^* such that $(g, g^*) \in \mathit{FRuleSet}$, P must be the winner of g^*. By hypothesis, all $cs(g^*)$ are not satisfiable. Therefore, $cs(g)$ is also not satisfiable by Theorem 1 (2).

☺

Theorem 4. *For all CK_n-formulas φ, all CK_n-models $\mathcal{M}$ and all worlds w of $\mathcal{M}$:*

$$\Vdash_{\mathcal{D}} \varphi \;\Rightarrow\; \mathcal{M}, w \models \varphi \quad .$$

Proof. To conclude: At the beginning of a game P states a hypothesis containing the formula φ_{H}. With this move the initial play g_1 is constructed. The constraint $(1, 0) : -\varphi_{\mathsf{H}}$ is then part of $cs(g_1)$. This means that for an interpretation $\mathcal{M}, \omega$, φ_{H} is not valid in $\omega(1, 0)$. We have shown that if P is the winner of the dialogue started in g_1 then $cs(g_1)$ is not satisfiable, i.e., the interpretation $\mathcal{M}, \omega$ does not exist. With this fact we can then conclude that if P wins with his hypothesis, φ_{H} must be a valid formula. ☺

References

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