Propositional Lax Logic

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M. Fairtlough, M. Mendler: Propositional Lax Logic. Information and Computation, Vol.137, No.1, 1997, pp. 1-33.

Proposed Running Head:

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Abstract

We investigate a peculiar intuitionistic modal logic, called Propositional Lax Logic (PLL), which has promising applications to the formal verification of computer hardware. The logic has emerged from an attempt to express correctness 'up to' behavioural constraints — a central notion in hardware verification — as a logical modality. As a modal logic it is special since it features a single modal operator $\bigcirc$ that has a flavour both of possibility and of necessity.

In the paper we provide the motivation for PLL and present several technical results. We investigate some of its proof-theoretic properties, presenting a cut-elimination theorem for a standard Gentzen-style sequent presentation of the logic. We go on to define a new class of fallible two-frame Kripke models for PLL. These models are unusual since they feature worlds with inconsistent information; furthermore, the only frame condition imposed is that the $\bigcirc$-frame be a subrelation of the $\supset$-frame. We give a natural translation of these models into Goldblatt's $\mathcal{J}$-space models of PLL. Our completeness theorem for these models yields a Gödel-style embedding of PLL into a classical bimodal theory of type (S4, S4) and underpins a simple proof of the finite model property. We proceed to prove soundness and completeness of several theories for specialized classes of models.

We conclude with a brief exploration of two concrete and rather natural types of model from hardware verification for which the modality $\bigcirc$ models correctness up to timing constraints. We obtain decidability of $\bigcirc$-free fragment of the logic of the first type of model, which coincides with the stable form of Maksimova's intermediate logic $L\Pi$.

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1 Introduction

The object of this paper is the rather curious modality $\bigcirc$ characterized by the axiom schemes

$$\begin{array}{lcl} \bigcirc R & : & M \supset \bigcirc M \\ \bigcirc M & : & \bigcirc\bigcirc M \supset \bigcirc M \\ \bigcirc S & : & (\bigcirc M \wedge \bigcirc N) \supset \bigcirc(M \wedge N) \end{array}$$

with the inference rule of Modus Ponens and the rule "from $M \supset N$ infer $\bigcirc M \supset \bigcirc N$". From a classical point of view the combination of these three axioms does not make much sense, however innocent each of these axioms may appear. Indeed, $\bigcirc$ has a flavour of both possibility and of necessity without being one or the other. Axioms $\bigcirc R$ and $\bigcirc M$ are typical of a modality of possibility $\Diamond$ while $\bigcirc S$ is typical for necessity $\Box$. On the other hand, in standard systems, say Lewis' modal system S4 [Chellas, 1980], the axiom $\bigcirc R$ is never adopted for necessity while $\bigcirc S$ never for possibility. In fact, if we add the axiom of the Excluded Middle (EM) and $\neg\bigcirc \mathit{false}$ (which is valid for both $\Diamond$ and $\Box$) to the modal system $\bigcirc R$, $\bigcirc M$, $\bigcirc S$ then $\bigcirc$ becomes trivial. We can derive both $\bigcirc M \supset M$ and $M \supset \bigcirc M$. In other words, there is no classical Kripke semantics for $\bigcirc$. In an intuitionistic setting, however, the situation is different. There, modal operators like $\bigcirc$ arise very naturally in various different ways and under various different names. In the following let us list some of them in order to motivate the interest in $\bigcirc$.

(1) Historically, the earliest appearance of an operator like $\bigcirc$ may have been in Curry's 1948 Notre Dame lectures on *A Theory of Formal Deducibility* published in [Curry, 1957]. These lectures contain some sketchy remarks on a modality endowed with axiom schemata, further refined in [Curry, 1952], that are essentially equivalent to the ones for $\bigcirc$.

(2) Reading implication as an ordering relation, the axioms and rules for $\bigcirc$ specify a class of monotone operators that arise in the study of the lattice-theoretic properties of topological spaces. Such operators were termed *nuclei* by Simmons [Simmons, 1978] and Macnab [Macnab, 1981]. The algebraic structure of nuclei can be generalized to the notion of a *modal operator* on a Heyting algebra [Macnab, 1981]. Goldblatt has shown that these algebras, which he calls *local (Heyting) algebras*, provide an appropriate algebraic semantics for intuitionistic propositional logic with a $\bigcirc$ modality [Goldblatt, 1981]. Goldblatt uses the term *geometric modality* for $\bigcirc$. The algebraic structure further features in category theory as a generalization of Grothendieck topologies. There the modal operator $\bigcirc$ on an Heyting algebra, usually referred to by the symbol j, becomes a *topology* on an elementary topos, and the local algebra becomes an *elementary site*. The interested reader is referred to [Goldblatt, 1979].

(3) The algebraic approach essentially characterizes the formal behaviour of $\bigcirc$ internally by the way it relates to implication $\supset$. However, when one is interested in $\bigcirc$ as a logical modality one expects instead to assign external meaning in terms of truth and validity. So, it is natural to try to extend the standard Kripke semantics for intuitionistic logic to encompass the modality as well. In [Goldblatt, 1981] two such classes of intuitionistic Kripke semantics, called $\mathcal{J}$-spaces and $\mathcal{J}$-frames, are presented. In these models an underlying Kripke frame is used to interpret the intuitionistic implication while the modality is

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interpreted by some extra data associated with the frame; in the first case this is a notion of neighbourhood and in the second case a notion of closeness of worlds. Both notions are conceived to give $\bigcirc M$ the meaning of "M is locally true".

(4) A different motivation for $\bigcirc$ can be drawn from general type theory. The formal properties of $\bigcirc$ viewed as an unary type constructor give precisely the data of a strong monad familiar from category theory. In fact, the propositions-as-types principle which yields an equivalence between the Intuitionistic Propositional Calculus (IPC) and bi-Cartesian closed categories can be extended to an equivalence between IPC extended by $\bigcirc$ and bi-Cartesian closed categories with a strong monad. This categorical structure is also known as the computational lambda calculus λ_c [Moggi, 1991]. Exploiting this connection strong monads have found their way into in functional programming, see *e.g.* their use in Haskell [Thompson, 1996]. The application of λ_c as a calculus of proofs has been investigated by Benton *et al.* [Benton et al., 1993], where the logic of $\bigcirc$ is called *computational logic* (CL).

(5) Our interest in the modality stems from a proof-theoretic interpretation of $\bigcirc$ introduced in [Mendler, 1990, Mendler, 1993]. It investigates an application to hardware verification in which the modality $\bigcirc$ formalizes the notion of correctness up to constraints. The corresponding calculi are called *Lax Logics*, where the term 'lax' is chosen to indicate the looseness associated with the notion of correctness up to constraints. The intuitive interpretation of $\bigcirc M$ is "for some constraint c, formula M holds under c". Clearly, different notions of constraint will have different properties, and thus will give rise to different axioms for $\bigcirc$. The generic interpretation leads to the three axioms $\bigcirc R$, $\bigcirc M$, and $\bigcirc S$. Axiom $\bigcirc R$ says "if M holds outright then it holds under a (trivial) constraint"; $\bigcirc M$ says "if under some constraint, M holds under another constraint, then M holds under an appropriately combined constraint"; finally, $\bigcirc S$ says "if M holds under a constraint, and N holds under a constraint, then the conjunction $M \wedge N$ holds under an appropriately combined constraint". This explains our use of the term *Propositional Lax Logic*, henceforth referred to as PLL, for the logic of $\bigcirc$.

(6) As a concrete instance of the constraint reading for $\bigcirc$ mentioned above (5), $\bigcirc$ can be applied to the timing analysis of combinational circuits. One can establish a direct correspondence between the axioms used in verifying the functional behaviour of a combinational circuit and the computation of data-dependent timing constraints: $\bigcirc R$ corresponds to a wire, which involves zero delay 0; $\bigcirc M$ deals with the sequential composition of circuits, which involves the addition $+$ of delays, and $\bigcirc S$ effects the parallel composition of circuits, which amounts to the maximum operation *max* on delays. In other words, by systematic translation of proofs in PLL into a term over the delay algebra $(\mathbf{Nat}, 0, +, max)$, we can extract verification-driven, and thus data-dependent, timing information. This is essentially an interpretation, in the sense of (4), in a concrete λ_c calculus. This idea has been worked out in [Mendler, 1996] for a fragment of the logic generated from atomic sentences and the derived implication M *leads to* $N \;=_{df}\; M \supset \bigcirc N$. Though the delay algebra $(\mathbf{Nat}, 0, +, max)$ may appear rather simple, it is sufficient for a large class of practical timing analyses for discrete dynamic systems [Baccelli et al., 1992].

The previous remarks indicate that however strange $\bigcirc$ may appear as a modality of logic it is a rather natural object well-known from other mathematical contexts. But while its

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algebraic and type-theoretic ramifications have been investigated its logical aspects seem to be largely unexplored.

This work stresses the logical view of $\bigcirc$ and introduces a novel and rather natural Kripke semantics for $\bigcirc$. The models, called *constraint models*, have two frame relations; one serves to realize the intuitionistic nature of the logic while the other is used to interpret the modality. Based on these models we give a full and faithful embedding of PLL into a classical bimodal theory of type (S4, S4) extending the well-known Gödel translation of intuitionistic logic into S4. This provides a classical explanation of $\bigcirc$ in terms of ordinary modalities.
We will use these constraint models towards a model-theoretic study of our reading of $\bigcirc$ as "under some constraint", which has been introduced previously only in a proof-theoretic sense. In this way we hope to convince the reader of an independent motivation of $\bigcirc$ from hardware verification. We will give two interesting subclasses of constraint models obtaining two concrete constraint interpretations of $\bigcirc$. These concrete models, which are related to (intermediate) intuitionistic logics introduced by Maksimova and Medvedev, verify that PLL has nontrivial expressiveness and illustrate the value of dropping Excluded Middle and $\neg\bigcirc \mathit{false}$ in concrete cases. We use the structure of the first model to establish the decidability of the stable form of Maksimova's logic and suggest applications of both models in hardware verification.

2 Propositional Lax Logic

The formulas of PLL are generated by the grammar

$$M \quad ::= \quad A \mid M \wedge M \mid M \vee M \mid M \supset M \mid \neg M \mid \bigcirc M$$

where A ranges over a countably infinite set of propositional constants pcs $= \{p_0, p_1, \ldots\}$. We will take $\equiv$ to abbreviate bi-implication and use the derived constants *true* and *false*. It is sometimes convenient to consider *false* as primitive and $\neg M$ as an abbreviation for $M \supset \mathit{false}$.
PLL is presented both as a Hilbert and as a Gentzen style calculus. The Hilbert system of PLL takes as axiom schemata all theorems of (or a complete set of axioms for) IPC, plus the modal axiom schemata $\bigcirc R$, $\bigcirc M$, $\bigcirc S$. The inference rules are Modus Ponens and the rule "from $M \supset N$ infer $\bigcirc M \supset \bigcirc N$". The finitary deduction relation induced by these axioms and rules is denoted by $\vdash_{\mathrm{PLL}}$. It is also possible to define PLL as a purely axiomatic extension of IPC.

Lemma 2.1 *$\Gamma \vdash_{\mathrm{PLL}} M$ iff M can be derived in* IPC *from Γ and the single axiom schema $(N \supset \bigcirc K) \equiv (\bigcirc N \supset \bigcirc K)$.*

Proof: Let $\vdash^+$ be the derivation relation obtained from IPC by adding the scheme $(N \supset \bigcirc K) \equiv (\bigcirc N \supset \bigcirc K)$. One shows that all instances of the three axioms $\bigcirc R$, $\bigcirc M$, $\bigcirc S$ can be derived in $\vdash^+$, and further that the rule "from $M \supset N$ infer $\bigcirc M \supset \bigcirc N$" is derivable in the strong form, namely, we have $\vdash^+ (M \supset N) \supset (\bigcirc M \supset \bigcirc N)$. In the

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other direction it suffices to show that all instances of $(N \supset \bigcirc K) \equiv (\bigcirc N \supset \bigcirc K)$ can be derived in $\vdash_{\mathrm{PLL}}$. Here $\vdash_{\mathrm{PLL}} (\bigcirc N \supset \bigcirc K) \supset (N \supset \bigcirc K)$ is a consequence of $\bigcirc R$, while for $\vdash_{\mathrm{PLL}} (N \supset \bigcirc K) \supset (\bigcirc N \supset \bigcirc K)$ one invokes all three axioms $\bigcirc R, \bigcirc M$, and $\bigcirc S$. Throughout the proof one makes use of the fact that all IPC theorems, in particular all substitution instances containing $\bigcirc$, are available. ∎

Proposition 2.2 (Deduction Theorem) $\Gamma, M \vdash_{\mathrm{PLL}} N$ *implies* $\Gamma \vdash_{\mathrm{PLL}} M \supset N$.

Proof: The statement follows immediately from the deduction theorem for IPC (see *e.g.* [Dummett, 1977]) and the fact that PLL is an axiomatic extension of IPC. ∎

The deduction theorem does not hold for the standard Hilbert presentation of ordinary modal logics. For instance in K, T, S4 [Chellas, 1980] we have $M \vdash \Box M$ but $\not\vdash M \supset \Box M$, and $M \supset N \vdash \Diamond M \supset \Diamond N$ but $\not\vdash (M \supset N) \supset (\Diamond M \supset \Diamond N)$.
The Gentzen-style calculus for PLL is presented in terms of ordinary *sequents* $\Gamma \vdash \Delta$, where Γ is a finite, possibly empty, list of *hypotheses* and Δ a finite list of *assertions* with length 0 or 1. The complete set of our sequent rules is listed in figure 1. The inference rules for deriving sequents are the standard ones for IPC plus two special rules $\bigcirc R$ and $\bigcirc L$ which capture the properties of $\bigcirc$:

$$\frac{\Gamma \vdash M}{\Gamma \vdash \bigcirc M}\ \bigcirc R \qquad\qquad \frac{\Gamma, M \vdash \bigcirc N}{\Gamma, \bigcirc M \vdash \bigcirc N}\ \bigcirc L.$$

These rules are the ones suggested by [Curry, 1957], and may be seen as a sequent-style version of the natural deduction system for $\bigcirc$ used in [Mendler, 1993]. The rules have independently been considered by [Benton et al., 1993]. There are other alternative formalizations of PLL, *e.g.* a tableau calculus has been investigated in [Avellone and Ferrari, 1996].

Theorem 2.3 *The Hilbert and Gentzen systems for* PLL *are equivalent, i.e. for all formulas* M, $\vdash_{\mathrm{PLL}} M$ *iff* $\vdash M$ *is derivable.*

Proof: One proves a stronger theorem, showing that when Γ is finite and Δ contains at most one formula, the sequent $\Gamma \vdash \Delta$ is derivable iff $\Gamma \vdash_{\mathrm{PLL}} \bigvee \Delta$, where $\bigvee \Delta = M$ if $\Delta = \{M\}$, and $\bigvee \emptyset = \mathit{false}$. Both directions can be established by induction on derivations. ∎

Theorem 2.4 (Strong Conservativity) *Let M be a theorem of* PLL. *Then the formula M', where M' is obtained from M by removing all occurrences of* $\bigcirc$, *is a theorem of* IPC.

Proof: By induction on the structure of derivations one shows that if $\Gamma \vdash M$ then $\Gamma' \vdash M'$. ∎

Another way of turning theorems of PLL into theorems of IPC is obtained by replacing all sub-formulas prefixed by $\bigcirc$ by *true*. Both results are special instances of the more general

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Logical Rules

$$\frac{\Gamma \vdash M \quad \Gamma \vdash N}{\Gamma \vdash M \wedge N} \wedge R \qquad \frac{\Gamma, M, N \vdash \Delta}{\Gamma, M \wedge N \vdash \Delta} \wedge L$$

$$\frac{\Gamma, M \vdash \Delta \quad \Gamma, N \vdash \Delta}{\Gamma, M \vee N \vdash \Delta} \vee L$$

$$\frac{\Gamma \vdash M}{\Gamma \vdash M \vee N} \vee R_1 \qquad \frac{\Gamma \vdash N}{\Gamma \vdash M \vee N} \vee R_2$$

$$\frac{\Gamma, M \vdash N}{\Gamma \vdash M \supset N} \supset R \qquad \frac{\Gamma \vdash M \quad \Gamma, N \vdash \Delta}{\Gamma, M \supset N \vdash \Delta} \supset L$$

$$\frac{\Gamma, M \vdash}{\Gamma \vdash \neg M} \neg R \qquad \frac{\Gamma \vdash M}{\Gamma, \neg M \vdash} \neg L$$

$$\frac{\Gamma \vdash M}{\Gamma \vdash \bigcirc M} \bigcirc R \qquad \frac{\Gamma, M \vdash \bigcirc N}{\Gamma, \bigcirc M \vdash \bigcirc N} \bigcirc L$$

Structural Rules

$$\frac{}{M \ \vdash \ M} \, id \qquad \frac{\Gamma \vdash M \quad \Gamma, M \vdash \Delta}{\Gamma \vdash \Delta} \, cut$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, M \vdash \Delta} \, weakL \qquad \frac{\Gamma \vdash}{\Gamma \vdash M} \, weakR$$

$$\frac{\Gamma, M, M \vdash \Delta}{\Gamma, M \vdash \Delta} \, contr \qquad \frac{\Gamma, M, N, \Gamma' \vdash \Delta}{\Gamma, N, M, \Gamma' \vdash \Delta} \, exch$$

Figure 1: Gentzen Rules for PLL.

result that the translation $\bigcirc M \equiv C \supset M$ preserves provability; for the first take $C \equiv true$ and for the second take $C \equiv false$. From the latter translation we may conclude, for instance, that $\neg \bigcirc false$ and (from the general translation) that $\bigcirc(M \vee N) \supset (\bigcirc M \vee \bigcirc N)$ are not theorems of PLL. This ensures that PLL is nontrivial extension of IPC, in the sense that it is not possible to transform a theorem of IPC into a theorem of PLL by arbitrarily introducing $\bigcirc$s.

Theorem 2.5 (Strong Extensionality) PLL *is strongly extensional, i.e. the scheme* $(M \equiv N) \supset (\mathcal{C}[M] \equiv \mathcal{C}[N])$ *is admissible, where* $\mathcal{C}[\_]$ *is an arbitrary syntactic context and* M, N *arbitrary formulas.*

Proof: The proof is by induction on the structure of $\mathcal{C}[\_]$. The interesting case, of course, is when $\mathcal{C}[\_] = \bigcirc[\_]$. But $\vdash (M \equiv N) \supset (\bigcirc M \equiv \bigcirc N)$ may be easily derived using rules

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$\bigcirc L$ and $\bigcirc R$ (of the Gentzen calculus). ∎

Theorem 2.6 (Cut Elimination) *If* $\vdash \Delta$ *is derivable, then it is derivable without the cut rule.*

Proof: The proof uses the same method that works for IPC [Dummett, 1977]. One new reduction step needs to be introduced, as shown in figure 2. Cut elimination has

$$\dfrac{\dfrac{\overset{\Pi_1}{\Gamma \vdash M}}{\Gamma \vdash \bigcirc M}\,\bigcirc R \qquad \dfrac{\overset{\Pi_2}{\Gamma, M \vdash \bigcirc N}}{\Gamma, \bigcirc M \vdash \bigcirc N}\,\bigcirc L}{\Gamma \vdash \bigcirc N}\,cut$$

$$\overset{reduce}{\Longrightarrow} \qquad \dfrac{\overset{\Pi_1}{\Gamma \vdash M} \qquad \overset{\Pi_2}{\Gamma, M \vdash \bigcirc N}}{\Gamma \vdash \bigcirc N.}\,cut$$

Figure 2: Additional Primitive Cut Reduction Step

independently been proven by [Benton et al., 1993]. ∎

Direct consequences of cut-elimination are the disjunction and the sub-formula property, and the admissibility of the rule "from $\bigcirc M$ infer M", which is the inverse of the necessitation rule of standard modal logics.

Lemma 2.7

(i) $\vdash_{\mathrm{PLL}} M \vee N$ *implies* $\vdash_{\mathrm{PLL}} M$ *or* $\vdash_{\mathrm{PLL}} N$

(ii) $\vdash_{\mathrm{PLL}} \bigcirc M$ *implies* $\vdash_{\mathrm{PLL}} M$

(iii) *If* $\Gamma \vdash \Delta$ *is derivable, then there exists a derivation which involves only sub-formulas of* Γ *and* Δ.

From the sub-formula property (iii) we get the decidability of PLL. This theorem is proven in [Goldblatt, 1981] by semantic methods.

Theorem 2.8 (Decidability) PLL *is decidable.*

We have seen that PLL combines a number of properties (in particular deduction theorem and the interpretation $\bigcirc M = \mathit{true}$) which are rather strong for a modal logic. Although from a formal point of view every unary syntactic operator may be called a 'modality' one wonders whether the proof-theoretic properties of $\bigcirc$ are not in fact too strong for it to be an interesting modality in a semantic sense. It turns out that $\bigcirc$ indeed can be given a proper and nontrivial semantics in terms of Kripke models. One necessary condition on a satisfactory notion of Kripke model, of course, is that is should explain the modality $\bigcirc$ in terms of a corresponding semantic accessibility relation. In the following section we present one such type of model.

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3 Constraint Models for PLL

Kripke-style analyzes have been given for other intuitionistic modal logics, for instance by Simpson [Simpson, 1994] and Plotkin and Stirling [Plotkin and Stirling, 1986] for system IK, by Fischer-Servi [Fischer-Servi, 1980] for the class of (∗)-IC systems, and by Ewald [Ewald, 1986] for an intuitionistic tense logic. The approach taken here most closely follows [Plotkin and Stirling, 1986] in using one set of worlds but two separate frame relations to interpret $\bigcirc$ and $\supset$. This satisfies our requirement that $\bigcirc$ be given a Kripke-style interpretation. As a result of our approach we obtain a full and faithful embedding of PLL into a classical bimodal (S4, S4) logic. This gives a classical account of PLL which extends the well-known Gödel embedding of IPC, and is different from the embedding of intuitionistic modal logics suggested by Fischer-Servi [Fischer-Servi, 1980]. A quite different kind of semantics was given by Goldblatt [Goldblatt, 1981], in which only the intuitionistic part $\supset$ is represented by a frame relation, while the modality is realized by some extra topological information on the intuitionistic frame.

Definition 3.1 (Kripke Constraint Model) *A (Kripke)* constraint model *for* PLL *is a quintuple* $\mathcal{C} = (W, R_m, R_i, V, F)$, *where* W *is a non-empty set,* R_m, R_i *are binary relations on* W, $F \subseteq W$, *and* V *is a map that assigns to every propositional constant* A *of* PLL *a subset* $V(A) \subseteq W$. *These data are subject to the following conditions:*

- R_m, R_i *are preorders, i.e. reflexive and transitive relations, and* $R_m \subseteq R_i$,
- F *and* V *are hereditary w.r.t.* R_i, *i.e. if* $w\ R_i\ v$, *then* $w \in F$ *implies* $v \in F$, *and* $w \in V(A)$ *implies* $v \in V(A)$,
- V *is full on* F, *i.e.* $F \subseteq V(A)$.

If $w\ R_m\ v$ then we say that v is a *constraining* of w, or v is reachable from w under a *constraint*. Elements of F are *fallible* worlds and if $w\ R_m\ v$ and $v \in F$, then intuitively the constraint leading to v is inconsistent with world w. Models with fallible worlds are not a new concept. They have been introduced previously to admit intuitionistic meta-theory for intuitionistic logic, see *e.g.* [Troelstra and van Dalen, 1988, Dummett, 1977]. As we will show later on, in our context, fallible worlds arise naturally from the constraint interpretation.

Definition 3.2 (Validity) *Let* $\mathcal{C} = (W, R_m, R_i, V, F)$ *be a constraint model for* PLL. *Given a formula* M *and* $w \in W$, M *is* valid *at* w *in* $\mathcal{C}$, *written* $\mathcal{C}, w \models M$ *iff*

- M *is a propositional constant* A *and* $w \in V(A)$;
- M *is* $N \wedge K$ *and both* $\mathcal{C}, w \models N$ *and* $\mathcal{C}, w \models K$;
- M *is* $N \vee K$ *and* $\mathcal{C}, w \models N$ *or* $\mathcal{C}, w \models K$;
- M *is true; or* M *is false and* $w \in F$;

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- *M is $N \supset K$ and for all $v \in W$ such that $w\ R_i\ v$, $\mathcal{C}, v \models N$ implies $\mathcal{C}, v \models K$;*
- *M is of form $\bigcirc N$ and for all $v \in W$, $w\ R_i\ v$, there exists $u \in W$ with $v\ R_m\ u$ such that $\mathcal{C}, u \models N$.*

A formula M is valid in $\mathcal{C}$*, written $\mathcal{C} \models M$, if for all $w \in W$, M is valid at w in $\mathcal{C}$; M is* valid*, written $\models M$, if M is valid in any constraint model $\mathcal{C}$.*

Disregarding the fallible worlds, for modal-free formulas validity is defined exactly as for intuitionistic logic on the underlying frame (W, R_i, V). Validity behaves as in ordinary intuitionistic logic, *viz.* it is hereditary with respect to the accessibility relations. Formally, if $w \models M$ and $w\ R_i\ v$, then $v \models M$. This is due to the transitivity of R_i. Since R_m is a subrelation of R_i, validity is hereditary with respect to R_m too. Worlds w, v with $w\ R_i\ v$ and $v\ R_i\ w$ validate the same formulas and can thus be identified. Hence, it is no restriction to assume that the relation R_i is a partial order, *i.e.* antisymmetric. Note that $\bigcirc$ is hereditary w.r.t. the intuitionistic frame R_i without further imposing a confluence frame condition as in the models for IK [Plotkin and Stirling, 1986].
Some remarks concerning our definition of validity are in order. Observe that the clause for validity of $\bigcirc N$ is a $\forall\exists$ statement. This endows $\bigcirc$ with properties of both possibility and of necessity. Secondly, one notes that fallible worlds validate all formulas and that $\neg \bigcirc \mathit{false}$ is not valid in general. Also, our semantics of $\bigcirc$ does not validate the scheme $\bigcirc(M \vee N) \supset \bigcirc M \vee \bigcirc N$, a fact that is important if the semantics is to capture the proof-theoretic properties of PLL. Both this scheme and $\neg \bigcirc \mathit{false}$ are generally adopted for modality $\Diamond$, even for intuitionistic logics such as IK and apparently also by the class $(*)$-IC of logics considered by Fischer-Servi in [Fischer-Servi, 1980]. We will present concrete constraint models falsifying as well as validating these axioms. Finally notice that there is no point in defining a 'necessity' modality, in contrast to IK. Its definition

$$w \models \Box M \quad \text{iff} \quad \forall v, u.\ w\ R_i\ v\ \ \&\ \ v\ R_m\ u \Rightarrow u \models M$$

yields nothing new because of the frame condition $R_m \subseteq R_i$.

Theorem 3.3 (Soundness) *If $\vdash_{\mathrm{PLL}} M$ then $\models M$.*

Proof: We lift the notion of validity to sequents in the following way: A sequent $\Gamma \vdash K$ is *valid in* model $\mathcal{C}$ if for all w, whenever all hypotheses $M \in \Gamma$ are valid at w in $\mathcal{C}$, then the assertion K is valid at w in $\mathcal{C}$; a sequent $\Gamma \vdash$ is valid in $\mathcal{C}$ if the only worlds at which all hypotheses $M \in \Gamma$ are valid in $\mathcal{C}$ are fallible worlds.
One then shows by induction on derivations that if $\Gamma \vdash \Delta$ is derivable then $\Gamma \vdash \Delta$ is valid in all models. The hereditariness of validity, and thus transitivity of R_i and inclusion $R_m \subseteq R_i$, is used for the rules $\supset R$, $\neg R$, $\bigcirc R$, and $\bigcirc L$. The reflexivity of R_i is used to show soundness of $\supset L$, $\neg L$, $\bigcirc L$. Finally, the reflexivity of R_m is exploited for $\bigcirc R$, and transitivity of R_m for $\bigcirc L$. ∎

Another type of models for PLL are the $\mathcal{J}$-frames and $\mathcal{J}$-spaces of Goldblatt [Goldblatt, 1981]. Just as in our work, these models are built on an intuitionistic frame

(W, R_i). However, they do not have fallible worlds F and in place of our modal frame relation R_m some topological structure on (W, R_i) is used. We will now give a rather natural semantics-preserving translation of constraint models into $\mathcal{J}$-spaces, that preserves the underlying intuitionistic frame. The other direction and the connection with $\mathcal{J}$-frames, which are not considered here, are left as open problems.
We first recall the definitions given in [Goldblatt, 1981]. An *intuitionistic Kripke model* (IKM) is a triple (W, R_i, V) where W is a nonempty set, R_i a partial ordering on W and V a valuation, *i.e.* an assignment of hereditary subsets of W to propositional constants.

Definition 3.4 *An* $\mathcal{J}$-space *is given by an IKM* $\mathcal{S} = (W, R_i, V)$ *together with a map* γ *that assigns to every* $w \in W$ *a collection* $\gamma(w) \subseteq 2^W$ *of* R_i*-hereditary subsets of* W*, with the following properties:*

(N1) $w\ R_i\ v$ *implies* $\gamma(w) \subseteq \gamma(v)$

(N2) $R \in \gamma(w)$ *and* $S \in \gamma(w)$ *implies* $R \cap S \in \gamma(w)$

(N3) $R \in \gamma(w)$ *and* S *a* R_i*-hereditary subset of* W *such that* $R \subseteq S$ *imply* $S \in \gamma(w)$

(N4) $[w) = \{ v \mid w\ R_i\ v \} \in \gamma(w)$

(N5) *For any* R_i*-hereditary subset* $S \subseteq W$*, if* $\{ v \mid S \in \gamma(v) \} \in \gamma(w)$*, then* $S \in \gamma(w)$

Strictly, in [Goldblatt, 1981] the term $\mathcal{J}$*-spaces* is applied only to the underlying structure (W, R_i, γ) not including the valuation V. For modal-free formulas validity on $\mathcal{J}$-spaces, denoted by $\models_s$, is defined just like that for intuitionistic logic, on the underlying IKM. Validity for formulas $\bigcirc M$ is given by the clause

$$w \models_s \bigcirc M \quad \text{iff} \quad \exists S \in \gamma(w).\ \forall v \in S.\ v \models_s N.$$

Remark: Condition (N1) is to ensure hereditariness of validity. (N2) deals with the axiom $\bigcirc M \wedge \bigcirc N \supset \bigcirc(M \wedge N)$, (N3) with the rule that $\vdash M \supset N$ entails $\vdash \bigcirc M \supset \bigcirc N$, (N4) is for the axiom $M \supset \bigcirc M$, and finally (N5) ensures validity of $\bigcirc\bigcirc M \supset \bigcirc M$.

Theorem 3.5 *Let* $\mathcal{C} = (W, R_i, R_m, F, V)$ *be a non-trivial constraint model (i.e, one where* $W \neq F$*) and let* $(W^0, {R_i}^0, V^0)$ *be the underlying non-fallible IKM obtained from* (W, R_i, V) *by restriction to the set* $W \setminus F$*. Then, there exists* γ *such that* $\mathcal{S} = (W^0, {R_i}^0, \gamma, V^0)$ *is a* $\mathcal{J}$*-space such that for all* M*,*

$$\mathcal{C} \models M \quad \textit{iff} \quad \mathcal{S} \models_s M.$$

Proof: A subset $S \subseteq W$ is called R_m*-cofinal* for $w \in W$ iff S is R_i-hereditary and for all $u \in W$ such that $w\ R_i\ u$, there exists a $v \in S$ with $u\ R_m\ v$. In other words, S is R_m-cofinal for w if from every R_i-reachable successor of w the set S is R_m-reachable. For all $w \in W^0$ we take $\gamma(w)$ to be the set of all R_m-cofinal sets for w, restricted to W^0. We leave it to the reader to check the properties (N1)–(N5) and preservation of truth, *i.e.* that for all $w \in W^0$, $\mathcal{C}, w \models M$ iff $\mathcal{S}, w \models_s M$. ∎

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4 Completeness

In this section we prove completeness of PLL with respect to Kripke constraint models. If we had a method of translating Goldblatt's $\mathcal{J}$-spaces into equivalent constraint models, then completeness for constraint models would follow immediately from the following theorem.

Theorem 4.1 (Goldblatt) $\vdash_{\mathrm{PLL}} M$ *iff* M *is valid on all* $\mathcal{J}$*-spaces.*

Rather than searching for such a translation of models we give a separate completeness proof. We will follow the standard idea of constructing a counter model for every formula that is not derivable. The counter model employs a suitable generalization of the Lindenbaum construction, in which worlds are triples

$$(\Gamma, \Delta, \Theta)$$

of sets of formulas, called *theories*, subject to an abstract consistency condition which reflects the semantic rôle of its components (*cf.* [Fitting, 1983]).
The model will be set up so that at a world $w = (\Gamma, \Delta, \Theta)$ the formulas in Γ are validated at w, the formulas in Δ are falsified at w, and the formulas in Θ are falsified at every world R_m-reachable from w. The sets Θ are a special feature of our completeness proof and of PLL. They are introduced to make up for the fact that falsity of a formula $\bigcirc M$ cannot be expressed by including M (or a sub-formula of M) in Γ or Δ. We need to keep track of these separately.
Another special feature of the proof is the notion of consistency. A theory (Γ, Δ, Θ) is *consistent* if for every choice of formulas $N_1, \ldots, N_n \in \Delta$, and $K_1, \ldots, K_k \in \Theta$, such that $n + k \geq 1$, it is *not* the case that

$$\Gamma \quad \vdash \quad N_1 \vee \cdots \vee N_n \ \vee \ \bigcirc(K_1 \vee \cdots \vee K_k)$$

This definition is somewhat weaker than one might expect as it excludes the case $k = n = 0$. The disjunction on the right must always be nonempty, with the effect that the theories $(\Gamma, \emptyset, \emptyset)$, for any choice of Γ, are consistent for trivial reasons. The point here is that we take the empty disjunction to be the empty formula rather than *false*.
A consistent theory is *maximally consistent* if there is no proper consistent extension, under component-wise subset ordering. For instance, the distinguished theory $(\perp, \emptyset, \emptyset)$, where $\perp$ denotes the set of all formulas, is maximally consistent. Observe that if (Γ, Δ, Θ) is maximally consistent, then *false* $\in \Gamma$ iff $\Delta = \Theta = \emptyset$.

Lemma 4.2

- *Every consistent theory has a maximally consistent extension.*
- *If* (Γ, Δ, Θ) *is a maximally consistent theory then the following properties hold:*
  - **(i)** Γ *is deductively closed*
  - **(ii)** *If* $M \vee N \in \Gamma$ *then* $M \in \Gamma$ *or* $N \in \Gamma$

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(iii) *If* $M \supset N \in \Gamma$ *then* $M \in \Delta$ *or* $N \in \Gamma$

(iv) *If* $M \vee N \in \Delta$ *then* $M \in \Delta$ *and* $N \in \Delta$

(v) *If* $M \wedge N \in \Delta$ *then* $M \in \Delta$ *or* $N \in \Delta$

(vi) $\Theta \subseteq \Delta$

(vii) $M \in \Gamma$ *iff* $M \notin \Delta$

Proof: Let (Γ, Δ, Θ) be a consistent theory. We obtain a maximally consistent extension $(\Gamma^*, \Delta^*, \Theta^*)$ in the usual way by enumerating all formulas

$$B_0, B_1, \ldots, B_n, B_{n+1}, \ldots$$

and by building up a hierarchy of consistent theories

$$(\Gamma_0, \Delta_0, \Theta_0) \subseteq (\Gamma_1, \Delta_1, \Theta_1) \subseteq \cdots \subseteq (\Gamma_n, \Delta_n, \Theta_n) \subseteq (\Gamma_{n+1}, \Delta_{n+1}, \Theta_{n+1}) \subseteq \cdots$$

starting with $(\Gamma_0, \Delta_0, \Theta_0) = (\Gamma, \Delta, \Theta)$ and such that $(\Gamma_{n+1}, \Delta_{n+1}, \Theta_{n+1}) = (\Gamma_n \cup \{B_n\}, \Delta_n, \Theta_n)$ if it is consistent, otherwise $(\Gamma_{n+1}, \Delta_{n+1}, \Theta_{n+1}) = (\Gamma_n, \Delta_n \cup \{B_n\}, \Theta_n \cup \{B_n\})$ if it is consistent, otherwise $(\Gamma_{n+1}, \Delta_{n+1}, \Theta_{n+1}) = (\Gamma_n, \Delta_n \cup \{B_n\}, \Theta_n)$. Then,

$$(\Gamma^*, \Delta^*, \Theta^*) \stackrel{df}{=} (\bigcup_{n\in\omega} \Gamma_n, \bigcup_{n\in\omega} \Delta_n, \bigcup_{n\in\omega} \Theta_n).$$

$(\Gamma^*, \Delta^*, \Theta^*)$ is a maximally consistent theory. Note, if $\Gamma \vdash$ *false*, then by consistency of (Γ, Δ, Θ) we must have $\Delta = \Theta = \emptyset$, in which case the above construction will produce the maximally consistent extension $(\bot, \emptyset, \emptyset)$.
The second part of the lemma is not hard to verify. It uses the properties of the sequent calculus for $\vdash$, in particular the following two derived rules

$$\frac{\Gamma \vdash \bigvee X \ \vee \ \bigcirc \bigvee Y}{\Gamma \vdash \bigvee X' \ \vee \ \bigcirc \bigvee Y'} \ X \subseteq X' \text{ and } Y \subseteq Y' \qquad\qquad \frac{\Gamma \vdash M \vee N \qquad \Gamma, M \vdash N}{\Gamma \vdash N}$$

In the first rule X', Y' are finite sets of formulas and $\bigvee\{M_1, \ldots, M_n\}$ abbreviates $M_1 \vee \cdots \vee M_n$. When Z is empty then the corresponding disjunct $\bigvee Z$ in the first rule is dropped. The second rule is derivable from the structural rules (in particular the *cut* rule), $\vee L$, and *id*, whereas the first one also involves $\vee R_1, \vee R_2, \bigcirc R$, and $\bigcirc L$. The application of both these derived rules, as well as the structural rules, will be referred to as "structural reasoning" in the following. The seven claims in the second part of the lemma are now handled as follows:

(i) If $\Gamma \vdash M$ and $M \notin \Gamma$, then by maximality $\Gamma, M \vdash \bigvee \Delta' \vee \bigcirc \bigvee \Theta'$ for some finite subsets $\Delta' \subseteq \Delta$ and $\Theta' \subseteq \Theta$. By structural reasoning this implies $\Gamma \vdash \bigvee \Delta' \vee \bigcirc \bigvee \Theta'$, contradicting the consistency of (Γ, Δ, Θ). The remaining cases follow a similar pattern.

(ii) If neither M nor N are members of Γ, then $(\Gamma \cup \{M\}, \Delta, \Theta)$ and $(\Gamma \cup \{N\}, \Delta, \Theta)$ are inconsistent, by maximality. Thus, for some $\Delta_M, \Delta_N \subseteq \Delta$ and $\Theta_M, \Theta_N \subseteq \Theta$, we get associated — let us call them the "maximality" — proofs for $\Gamma, M \vdash \bigvee \Delta_M \;\vee\; \bigcirc \bigvee \Theta_M$ and $\Gamma, N \vdash \bigvee \Delta_N \;\vee\; \bigcirc \bigvee \Theta_N$. Applying structural reasoning and the $\vee L$ rule to the associated maximality proofs, we obtain the inconsistency of $(\Gamma \cup \{M \vee N\}, \Delta, \Theta)$, and hence $M \vee N \notin \Gamma$.

(iii) If $M \notin \Delta$ and $N \notin \Gamma$ we apply structural reasoning and the $\supset L$ rule to the associated maximality proofs to establish the inconsistency of $(\Gamma \cup \{M \supset N\}, \Delta, \Theta)$.

(iv) If $M \notin \Delta$ or $N \notin \Delta$, we may apply structural reasoning and $\vee R_1$ or $\vee R_2$ to the associated maximality proofs to establish the inconsistency of $(\Gamma, \Delta \cup \{M \vee N\}, \Theta)$; we might boil this argument down to the even more compact formulation "by maximality and $\vee R_1$";

(v) follows by maximality and $\wedge R$;

(vi) follows by maximality and the theorem $K \vee \bigcirc L \supset \bigcirc(K \vee L)$;

(vii) follows by maximality.

∎

We can now proceed to define a generic Kripke constraint model

$$\mathcal{C}^* = (W^*, R_m^*, R_i^*, V^*, F^*)$$

which falsifies all unprovable formulas. As the elements in W^* we take the maximally consistent theories $\mathcal{T} = (\Gamma, \Delta, \Theta)$. The accessibility relation R_i^* is simply the subset relation on the first component, *i.e.*

$$(\Gamma, \Delta, \Theta)\; R_i^*\; (\Gamma', \Delta', \Theta') \quad \stackrel{df}{\equiv} \quad \Gamma \subseteq \Gamma'$$

and constraint accessibility R_m^* is the subset relation in the first *and* third component:

$$(\Gamma, \Delta, \Theta)\; R_m^*\; (\Gamma', \Delta', \Theta') \quad \stackrel{df}{\equiv} \quad \Gamma \subseteq \Gamma' \quad \& \quad \Theta \subseteq \Theta'.$$

Valuation V^* and fallible nodes F^* are defined such that

$$\begin{aligned} V^*(A) \quad &\stackrel{df}{\equiv} \quad \{ (\Gamma, \Delta, \Theta) \mid A \in \Gamma \} \\ F^* \quad &\stackrel{df}{\equiv} \quad \{(\bot, \emptyset, \emptyset)\}. \end{aligned}$$

It is not hard to verify that these data indeed constitute a constraint Kripke model. The following properties make $\mathcal{C}^*$ a canonical model for PLL:

Lemma 4.3 *Let $\mathcal{T} = (\Gamma, \Delta, \Theta)$ be a maximally consistent theory. Then,*

- $M \in \Gamma$ *implies* $\mathcal{T} \models M$

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- $M \in \Delta$ *implies* $\mathcal{T} \not\models M$
- $M \in \Theta$ *implies that for all* $\mathcal{T}'$ *such that* $\mathcal{T}\ R^*_m\ \mathcal{T}'$, $\mathcal{T}' \not\models M$.

Proof: The lemma is proven by induction on the formula M. Here only the cases $M \equiv \bigcirc N$ and $M \equiv N \supset K$ will be treated, as they are the ones that drive the model along R^*_i and R^*_m. All other cases are achieved 'on-the-spot' using lemma 4.2.
It will be convenient to express the consistency condition for theories (Γ, Δ, Θ) in more concise but less precise form as

$$\Gamma \not\vdash_{\mathrm{PLL}} \bigvee \Delta \ \vee\ \bigcirc \bigvee \Theta,$$

noting that if the right hand side is the empty formula then the statement is trivially true.
• Suppose $\bigcirc N \in \Gamma$ and $\mathcal{T}_1$ is such that $\mathcal{T}\ R^*_i\ \mathcal{T}_1$. Then $\mathcal{T}_1 = (\Gamma_1, \Delta_1, \Theta_1)$ and $\Gamma \subseteq \Gamma_1$. We consider the theory $(\Gamma_1 \cup \{N\}, \emptyset, \Theta_1)$. We claim that this theory is consistent. Assume otherwise, then we must have $\Gamma_1, N \vdash_{\mathrm{PLL}} \bigcirc \bigvee \Theta_1$ and further by the deduction theorem, $\Gamma_1 \vdash_{\mathrm{PLL}} N \supset \bigcirc \bigvee \Theta_1$. Since we can prove

$$(N \supset \bigcirc \bigvee \Theta_1) \supset (\bigcirc N \supset \bigcirc \bigvee \Theta_1)$$

in PLL (by lemma 2.1) we conclude that $\Gamma_1, \bigcirc N \vdash_{\mathrm{PLL}} \bigcirc \bigvee \Theta_1$. But since $\bigcirc N \in \Gamma \subseteq \Gamma_1$ this contradicts the consistency of $\mathcal{T}_1$. By lemma 4.2 we can now find a maximally consistent extension $\mathcal{T}' = (\Gamma', \Delta', \Theta')$ of $(\Gamma_1 \cup \{N\}, \emptyset, \Theta_1)$. By definition, $\mathcal{T}_1\ R^*_m\ \mathcal{T}'$, and by the induction hypothesis on N, $\mathcal{T}' \models N$. Thus we have $\mathcal{T} \models \bigcirc N$.
• Suppose $\bigcirc N \in \Delta$. Consider the theory $(\Gamma, \emptyset, \{N\})$, which must be consistent for otherwise $\Gamma \vdash_{\mathrm{PLL}} \bigcirc N$, which contradicts consistency of $\mathcal{T}$. Now take a maximally consistent extension $\mathcal{T}' = (\Gamma', \Delta', \Theta')$ of $(\Gamma, \emptyset, \{N\})$. We claim that for all $\mathcal{T}_1$, $\mathcal{T}'\ R^*_m\ \mathcal{T}_1$, $\mathcal{T}_1 \not\models N$. Let $\mathcal{T}_1 = (\Gamma_1, \Delta_1, \Theta_1)$. By construction of $\mathcal{T}'$ and definition of R^*_m, $N \in \Theta' \subseteq \Theta_1$. By induction hypothesis on N, $\mathcal{T}_1 \not\models N$. This completes the proof that $\mathcal{T} \not\models \bigcirc N$.
• Suppose $N \supset K \in \Gamma$ and $\mathcal{T}_1 = (\Gamma_1, \Delta_1, \Theta_1)$ such that $\mathcal{T}\ R^*_i\ \mathcal{T}_1$. By definition of R^*_i, $N \supset K \in \Gamma \subseteq \Gamma_1$. By lemma 4.2 (iii) we have $N \in \Delta_1$ or $K \in \Gamma_1$. By induction hypothesis we infer that if $\mathcal{T}_1 \models N$ then $\mathcal{T}_1 \models K$. Thus, $\mathcal{T} \models N \supset K$.
• Suppose $N \supset K \in \Delta$. Consider the theory $(\Gamma \cup \{N\}, \{K\}, \emptyset)$. It must be consistent since otherwise $\Gamma, N \vdash_{\mathrm{PLL}} K$, whence by the deduction theorem $\Gamma \vdash_{\mathrm{PLL}} N \supset K$ which contradicts consistency of $\mathcal{T}$. Now take a maximally consistent extension $\mathcal{T}' = (\Gamma', \Delta', \Theta')$ of $(\Gamma \cup \{N\}, \{K\}, \emptyset)$. We have $\mathcal{T}\ R^*_i\ \mathcal{T}'$, $N \in \Gamma'$, and $K \in \Delta'$. By induction hypothesis, $\mathcal{T}' \models N$ and $\mathcal{T}' \not\models K$. But this means $\mathcal{T} \not\models N \supset K$.
• To prove the last statement of the lemma the cases $M \in \Theta$ are all treated in the same way: suppose $M \in \Theta$ and $\mathcal{T}_1 = (\Gamma_1, \Delta_1, \Theta_1)$ such that $\mathcal{T}\ R^*_m\ \mathcal{T}_1$. By definition of R^*_m, and the properties of maximally consistent theories, lemma 4.2 (vi), $M \in \Theta \subset \Theta_1 \subset \Delta_1$. Thus, we can appeal to the proofs above to conclude $\mathcal{T}_1 \not\models M$. ∎

Theorem 4.4 (Completeness) *If* $\models M$ *then* $\vdash_{\mathrm{PLL}} M$.

Proof: Suppose $\not\vdash_{\mathrm{PLL}} M$. Then $(\emptyset, \{M\}, \emptyset)$ is consistent. By lemma 4.2 there is a maximally consistent extension $\mathcal{T}$, and by lemma 4.3 $\mathcal{T} \not\models M$ in the constraint Kripke model $\mathcal{C}^*$. ∎

Three examples of counter models, one falsifying $\neg \bigcirc \mathit{false}$, one falsifying $\bigcirc(A \vee B) \supset (\bigcirc A \vee \bigcirc B)$ and one falsifying $\models (\bigcirc A \supset \bigcirc B) \supset \bigcirc(A \supset B)$, are shown in figure 3, where the dashed arrows represent R_i and the solid arrows R_m.

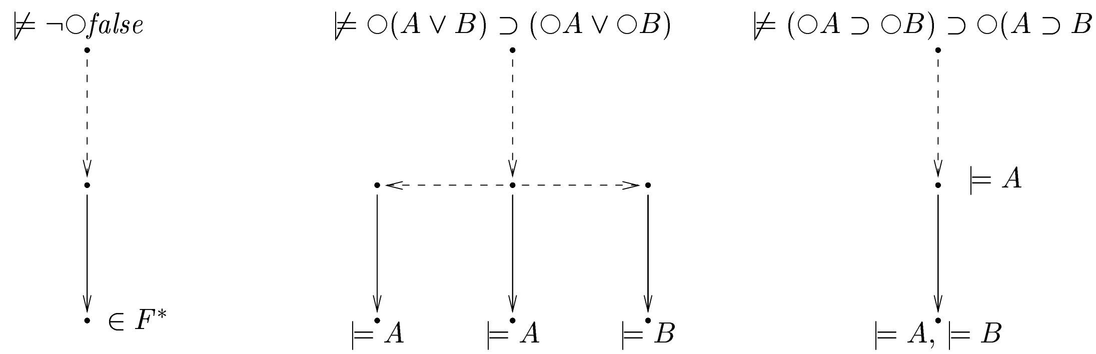

Figure 3: Three Counter Models

In section 6 we will discuss special cases of concrete constraint models validating the axiom schemes $\neg \bigcirc \mathit{false}$ and $\bigcirc(M \vee N) \supset (\bigcirc M \vee \bigcirc N)$. It turns out that these classes can be characterized as follows:

Theorem 4.5

- PLL $+ \neg \bigcirc \mathit{false}$ *is sound and complete for the class of constraint models with* $F = \emptyset$.
- PLL $+ \bigcirc(M \vee N) \supset (\bigcirc M \vee \bigcirc N)$ *is sound and complete for the class of constraint models where* R_m *and* R_i *are* mutually confluent*, i.e. if* $x\ R_m\ w$ *and* $x\ R_i\ v$*, then there exists* u *such that* $w\ R_i\ u$ *and* $v\ R_m\ u$*.*

Proof: Soundness of $\neg \bigcirc \mathit{false}$ is obvious if $F = \emptyset$. Soundness of the second axiom perhaps is not so obvious. For mutually confluent frame relations one first proves by induction on the structure of M that for all worlds w,

$$w \models \bigcirc M \quad \text{iff} \quad \exists u.\ w\ R_m\ u\ \ \&\ \ u \models M.$$

From this soundness of $\bigcirc(M \vee N) \supset (\bigcirc M \vee \bigcirc N)$ then follows directly.
The proof of completeness in both cases is obtained by simple specialization of the completeness proof for PLL (theorem 4.4).
• Suppose M is not derivable in PLL $+ \neg \bigcirc \mathit{false}$. Then the theory $(\{\neg \bigcirc \mathit{false}\}, \{M\}, \emptyset)$ is consistent and thus we can find a maximally consistent theory $\mathcal{T} = (\Gamma, \Delta, \Theta)$ so that $\neg \bigcirc \mathit{false} \in \Gamma$ and $M \in \Delta$. We know that $\mathcal{T} \not\models M$ in the sub-model of $\mathcal{C}^*$ generated by all theories $\mathcal{T}'$ such that $\mathcal{T}\ R_i^*\ \mathcal{T}'$. Though being a counter model for M it does contain

the fallible theory $(\bot, \emptyset, \emptyset)$ and thus is not of the desired form. However, one can show that by throwing out $(\bot, \emptyset, \emptyset)$ from the (counter) model we do not change validity of any formula. A sufficient condition for this is that $(\bot, \emptyset, \emptyset)$ cannot be accessed by R_m^*. In fact, one shows that $\neg \bigcirc \mathit{false} \in \Gamma$ and $(\Gamma, \Delta, \Theta)\ R_m^*\ (\Gamma', \Delta', \Theta')$ implies $\mathit{false} \notin \Gamma'$. For assume otherwise, then $(\Gamma', \Delta', \Theta') = (\bot, \emptyset, \emptyset)$ whence by definition of R_m^* we have $\Theta = \emptyset$. Since (Γ, Δ, Θ) is maximally consistent this implies that $\Gamma \vdash_{\mathrm{PLL}} \bigvee \Delta\ \vee\ \bigcirc \mathit{false}$ since the proper extension $(\Gamma, \Delta, \{\mathit{false}\})$ cannot be consistent. Now we use the assumption that $\neg \bigcirc \mathit{false} \in \Gamma$ (and the properties of deduction in the logic) to conclude that $\Gamma \vdash_{\mathrm{PLL}} \bigvee \Delta$ which contradicts the consistency of (Γ, Δ, Θ). Thus, from the assumption that M is not a theorem of $\mathrm{PLL} + \neg \bigcirc \mathit{false}$ we can construct a model without fallible nodes in which M is falsified. This proves the completeness statement for $\mathrm{PLL} + \neg \bigcirc \mathit{false}$.

• Suppose that M is not derivable in $\mathrm{PLL} + \bigcirc(X \vee Y) \supset (\bigcirc X \vee \bigcirc Y)$. Again we consider the canonical counter model generated by a maximally consistent theory $\mathcal{T} = (\Gamma, \Delta, \Theta)$ such that $M \in \Delta$ and such that Γ contains all substitution instances of the axiom scheme $\bigcirc(X \vee Y) \supset (\bigcirc X \vee \bigcirc Y)$. We are done if we can show that in the sub-model given by the maximally consistent theories $(\Gamma', \Delta', \Theta')$ with $\Gamma \subseteq \Gamma'$, R_i^* and R_m^* are mutually confluent. The first step is to observe that both Δ' and Θ' are uniquely determined by Γ' as follows:

$$\Delta' = \complement(\Gamma') \quad \text{and} \quad \Theta' = \delta\Delta',$$

where $\complement(\Gamma')$ is the complement of Γ' and $\delta\Delta' = \{ N \mid \bigcirc N \in \Delta' \}$. The first part was proven already in lemma 4.2. The second part is a consequence of the extra axioms in Γ' and seen as follows. Let $K \in \delta\Delta'$, *i.e.* $\bigcirc K \in \Delta'$, but $K \notin \Theta'$. Then, since $(\Gamma', \Delta', \Theta')$ is maximally consistent, we get $\Gamma' \vdash_{\mathrm{PLL}} \bigvee \Delta'\ \vee\ \bigcirc(\bigvee \Theta' \vee K)$. Now, by assumption, Γ' contains the axiom $\bigcirc(\bigvee \Theta' \vee K) \supset \bigcirc \bigvee \Theta' \vee \bigcirc K$, whence from both facts together we get $\Gamma' \vdash_{\mathrm{PLL}} \bigvee \Delta'\ \vee\ \bigcirc K\ \vee\ \bigcirc \bigvee \Theta'$ which contradicts consistency of $(\Gamma', \Delta', \Theta')$. Thus, $\delta\Delta' \subseteq \Theta'$. The other direction is obtained similarly, using the fact that $(\bigcirc N \vee \bigcirc K) \supset \bigcirc(N \vee K)$ is derivable in PLL.

The second step is to observe that $(\Gamma', \Delta', \Theta')\ R_m^*\ (\Gamma'', \Delta'', \Theta'')$ is equivalent to the condition $\Gamma' \subseteq \Gamma'' \subseteq \delta\Gamma'$. Assume $\Gamma' \subseteq \Gamma'' \subseteq \delta\Gamma'$. Then $\Theta' = \delta\Delta' = \delta(\complement(\Gamma')) = \delta\delta(\complement(\Gamma')) = \delta(\complement(\delta\Gamma')) \subseteq \delta(\complement(\Gamma'')) = \delta\Delta'' = \Theta''$, where the equation $\delta\delta X = \delta X$ holds generally for all deductively closed sets X, by virtue of the rule $\bigcirc M$. Thus, $(\Gamma', \Delta', \Theta')\ R_m^*\ (\Gamma'', \Delta'', \Theta'')$. Vice versa, if $(\Gamma', \Delta', \Theta')\ R_m^*\ (\Gamma'', \Delta'', \Theta'')$ we have $\Gamma' \subseteq \Gamma''$ and $\Gamma'' \subseteq \complement(\Theta'') \subseteq \complement(\Theta') = \complement(\delta\Delta') = \delta(\complement(\Delta')) = \delta\Gamma'$.

Now we can prove mutual confluence. Suppose we are given three maximally consistent theories $(\Gamma', \Delta', \Theta')$, $(\Gamma_1, \Delta_1, \Theta_1)$, and $(\Gamma_2, \Delta_2, \Theta_2)$ such that $\Gamma \subseteq \Gamma' \cap \Gamma_1 \cap \Gamma_2$ and such that

$$(\Gamma', \Delta', \Theta')\ R_i^*\ (\Gamma_1, \Delta_1, \Theta_1) \quad \text{and} \quad (\Gamma', \Delta', \Theta')\ R_m^*\ (\Gamma_2, \Delta_2, \Theta_2).$$

We need to find a $\mathcal{T}'$ such that $(\Gamma_1, \Delta_1, \Theta_1)\ R_m^*\ \mathcal{T}'$ and $(\Gamma_2, \Delta_2, \Theta_2)\ R_i^*\ \mathcal{T}'$. We claim that any maximally consistent extension $\mathcal{T}'$ of the theory $(\delta\Gamma_1, \emptyset, \Theta_1)$ will do. For such a $\mathcal{T}'$ to exist $(\delta\Gamma_1, \emptyset, \Theta_1)$ must be consistent. Suppose it is not, then $\delta\Gamma_1 \vdash_{\mathrm{PLL}} \bigcirc \bigvee \Theta_1$ which, by the properties of the logic, implies that $\Gamma_1 \vdash_{\mathrm{PLL}} \bigcirc \bigvee \Theta_1$ contradicting consistency of $(\Gamma_1, \Delta_1, \Theta_1)$. Thus, let $\mathcal{T}' = (\Gamma_3, \Delta_3, \Theta_3)$ be a maximally consistent extension of

$(\delta\Gamma_1, \emptyset, \Theta_1)$. One verifies $\Gamma_1 \subseteq \delta\Gamma_1 \subseteq \Gamma_3$ and $\Theta_1 \subseteq \Theta_3$, hence $(\Gamma_1, \Delta_1, \Theta_1)\ R_m^*\ \mathcal{T}'$ as desired. Further, $\Gamma_2 \subseteq \delta\Gamma'$ by the second observation above and thus $\Gamma_2 \subseteq \delta\Gamma' \subseteq \delta\Gamma_1 \subseteq \Gamma_3$. Thus, $(\Gamma_2, \Delta_2, \Theta_2)\ R_i^*\ \mathcal{T}'$ which completes the proof that the presence of the axioms $\bigcirc(X \vee Y) \supset (\bigcirc X \vee \bigcirc Y)$ forces R_m^* and R_i^* in the canonical model to be mutually confluent. ∎

Our proof of completeness is classical, *i.e.* nonconstructive. It does not yield an effective method of constructing a counter model for unprovable sequents. However, from the work of Avellone and Ferrari [Avellone and Ferrari, 1996], which uses a different, tableau-based presentation of PLL it is clear that a constructive proof of completeness for our constraint models is possible. In fact, PLL has the finite model property for our class of constraint models.

Theorem 4.6 (Finite Model Property) $\vdash_{\text{PLL}} M$ *iff* $\mathcal{C} \models M$ *for all finite constraint models* $\mathcal{C}$.

Proof: Soundness is obvious. Completeness hinges on the fact that, as in intuitionistic logic, the validity or refutation of a formula M at a given world w of a constraint model only depends on the validity or refutation of all of its proper subformulas at w and at all v that are R_i-reachable from w. So, at each world only a finite amount of information is relevant for M. Using this one can devise a suitable quotient (filtration) of a given counter model for M, that preserves the refutation of M but has only a finite number of elements.

Concretely, let $Sf(M)$ be the set of subformulas of M (we consider *false* as a subformula of every formula), and $\mathcal{C} = (W, R_i, R_m, V, F)$ a refutation model for M. In our constraint models two kinds of information are relevant of a given world w. Firstly, as in the intuitionistic case, we need to preserve the set $T(w)$ of subformulas that are validated at w, *i.e.* the set

$$T(w) \quad := \quad \{ N \in Sf(M) \mid w \models N \}.$$

Secondly, we need to preserve the set of subformulas that are refuted on all R_m-reachable successors of w, *i.e.* the set

$$F_m(w) \quad := \quad \{ N \in Sf(M) \mid \forall v.\ w\ R_m\ v \ \Rightarrow\ v \not\models N \}.$$

This part of the information captures the semantic behaviour of the modality $\bigcirc$. We then define an equivalence relation on W as follows:

$$w \equiv v \quad \text{iff} \quad T(w) = T(v) \quad \& \quad F_m(w) = F_m(v).$$

Since $Sf(M)$ is finite it is clear that there are only a finite number of equivalence classes $[w]_\equiv$. We now define the filtration model

$$\mathcal{C}_\equiv \quad = \quad (W|_\equiv, R_i|_\equiv, R_m|_\equiv, V|_\equiv, F|_\equiv)$$

over the set $W|_\equiv$ of equivalence classes, by stipulating $[w]_\equiv\ R_i|_\equiv\ [v]_\equiv$ iff $T(w) \subseteq T(v)$; $[w]_\equiv\ R_m|_\equiv\ [v]_\equiv$ iff $T(w) \subseteq T(v)$ and $F_m(w) \subseteq F_m(v)$; $[w]_\equiv \in V_\equiv(A)$ iff $A \notin Sf(M)$ or

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$w \in V(A)$; $[w]_\equiv \in F|_\equiv$ iff $w \in F$. One verifies that this construction yields a well-defined finite constraint model that validates exactly the same M-subformulas as $\mathcal{C}$. Thus, if $\not\vdash_{\mathrm{PLL}} M$ we can apply this filtration to the canonical counter model $\mathcal{C}^*$ constructed in the proof of the completeness theorem 4.4 to get a finite counter model $\mathcal{C}^*|_\equiv$ for M. ∎

5 Embedding of PLL in Classical Modal Logic

It is well-known that intuitionistic logic can be encoded in the classical modal logic S4, using Gödel's translation [Gödel, 1932]. In fact, the completeness of intuitionistic logic for the standard intuitionistic Kripke semantics can be seen as a corollary of the faithfulness of Gödel's translation. The main result of this section is to show that for the intuitionistic modal logic PLL too a faithful translation into classical modal logic can be obtained from the Kripke semantics presented in the previous section. We shall embed PLL into a classical bimodal theory of type (S4, S4).

Classical bimodal logic has the usual propositional connectives together with two dual pairs of modalities $\Box_i, \Diamond_i, \Box_m, \Diamond_m$. A bimodal *model* is a Kripke structure $\mathcal{M} = (W, R_m, R_i, V)$ where W is a nonempty set, R_i, R_m are binary relations on W, and V is a map that assigns to every propositional constant A a subset $V(A) \subseteq W$. The notion of validity in bimodal models is as usual and assumed to be understood (see *e.g.* [Popkorn, 1994]).
A bimodal logic of type (S4, S4) has as axioms the standard propositional ones plus the modal schemes

$$\begin{array}{rcl@{\qquad}rcl} T_i & : & \Box_i M \supset M & T_m & : & \Box_m M \supset M \\ 4_i & : & \Box_i M \supset \Box_i \Box_i M & 4_m & : & \Box_m M \supset \Box_m \Box_m M \\ K_i & : & \Box_i (M \supset N) \supset \Box_i M \supset \Box_i N & K_m & : & \Box_m (M \supset N) \supset \Box_m M \supset \Box_m N \end{array}$$

and Modus Ponens together with necessitation

$$\vdash M \;\Rightarrow\; \vdash \Box_i M \qquad\qquad \vdash M \;\Rightarrow\; \vdash \Box_m M$$

as rules of inference. As usual the necessity modalities $\Box_i$, $\Box_m$ are taken as primitive and the possibilities are introduced as their classical duals, *i.e.* $\Diamond_i M = \neg\Box_i\neg M$ and $\Diamond_m M = \neg\Box_m\neg M$. The bimodal theory we are interested in is obtained from bimodal logic of type (S4, S4) by adding the axiom scheme

$$Sub \;:\; \Box_i M \supset \Box_m M.$$

The resulting theory we denote by [S4, S4], where the square brackets are meant to indicate the presence of the axiom *Sub*. A [S4, S4]-*model* is a bimodal model $\mathcal{M} = (W, R_m, R_i, V)$ where R_i, R_m are reflexive, transitive, and satisfy $R_m \subseteq R_i$. It is straightforward to show from results in [Popkorn, 1994] that the theory [S4, S4] is sound and (Kripke) complete for the class of [S4, S4]-models.

Let f be a distinguished propositional constant in the following. We translate every formula M of PLL into a bimodal formula M^g as follows:

$$false^g \;=\; \Box_i f$$

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$$
\begin{aligned}
A^g &= \Box_i(A \vee f) \\
(M \wedge N)^g &= M^g \wedge N^g \\
(M \vee N)^g &= M^g \vee N^g \\
(M \supset N)^g &= \Box_i(M^g \supset N^g) \\
(\bigcirc M)^g &= \Box_i \Diamond_m M^g,
\end{aligned}
$$

where A ranges over propositional constants.

Theorem 5.1 *Let M be a formula of* PLL *that does not contain the propositional constant f. Then, $\vdash_{\mathrm{PLL}} M$ iff* [S4, S4] $\vdash M^g$.

Proof: The theorem is a direct consequence of soundness and completeness of the respective logics, and the close relationship between their models. There is a natural way to translate both types of models into each other preserving the validity of formulas. All we need to do is to translate the valuation part, the bimodal structure remains the same.

($\Rightarrow$) Let $\mathcal{M} = (W, R_m, R_i, V)$ be a [S4, S4]-model and $\mathcal{M}_g = (W, R_m, R_i, V_g, F_g)$ the induced Kripke constraint model with $V_g(A) = \{ w \in W \mid \forall v \in W.\ w\ R_i\ v \Rightarrow v \in V(A) \cup V(f) \}$ and $F_g = V_g(f)$. We prove by structural induction that for all formulas M of PLL that do not contain f,

$$\mathcal{M}, w \models M^g \quad \Leftrightarrow \quad \mathcal{M}_g, w \models M,$$

where $\models$ on the left is classical validity in [S4, S4]-models, while $\models$ on the right is intuitionistic validity in constraint models. From this it follows that if M is valid in all constraint models then M^g is valid in all [S4, S4]-models. Hence, by completeness of [S4, S4], $\vdash_{\mathrm{PLL}} M$ implies [S4, S4] $\vdash M^g$.

• $\mathcal{M}, w \models \mathit{false}^g$ iff $\mathcal{M}, w \models \Box_i f$ iff $\forall v.\ w\ R_i\ v \Rightarrow \mathcal{M}, v \models f$, iff $w \in V_g(f)$ iff $\mathcal{M}_g, w \models \mathit{false}$.

• $\mathcal{M}, w \models A^g$ iff $\mathcal{M}, w \models \Box_i(A \vee f)$ iff $w \in V_g(A)$ iff $\mathcal{M}_g, w \models A$.

• Conjunction $M \wedge N$ and disjunction $M \vee N$ present no difficulties.

• $\mathcal{M}, w \models (M \supset N)^g$ iff $\mathcal{M}, w \models \Box_i(M^g \supset N^g)$. This is equivalent to the statement that for all v with $w\ R_i\ v$, $\mathcal{M}, v \models M^g$ implies $\mathcal{M}, v \models N^g$. By induction hypothesis this is equivalent to $\mathcal{M}_g, v \models M$ implies $\mathcal{M}_g, v \models N$. Hence, $\mathcal{M}, w \models (M \supset N)^g$ is equivalent to $\mathcal{M}_g, w \models M \supset N$.

• $\mathcal{M}, w \models (\bigcirc M)^g$ iff $\mathcal{M}, w \models \Box_i \Diamond_m M^g$. Using the induction hypothesis for M, this is readily seen to be the same as the statement $\mathcal{M}_g, w \models \bigcirc M$.

($\Leftarrow$) Let $\mathcal{C} = (W, R_m, R_i, V, F)$ be a constraint model and $\mathcal{C}^g = (W, R_m, R_i, V^g)$ the induced [S4, S4]-model obtained by putting $V^g(A) = V(A)$ if $A \neq f$ and $V^g(f) = F$. We claim that for all formulas M that do not contain f,

$$\mathcal{C}, w \models M \quad \Leftrightarrow \quad \mathcal{C}^g, w \models M^g.$$