Intensional completeness in an extension of Gödel/Dummett logic*

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Abstract

We enrich intuitionistic logic with a lax modal operator $\bigcirc$ and define a corresponding intensional enrichment of Kripke models $M = (W, \sqsubseteq, V)$ by a function T giving an effort measure $T(w, u) \in \mathbb{N} \cup \{\infty\}$ for each $\sqsubseteq$-related pair (w, u). We show that $\bigcirc$ embodies the abstraction involved in passing from "φ true up to bounded effort" to "φ true outright". We then introduce a refined notion of **intensional validity** $M \models p : \varphi$ and present a corresponding intensional calculus iLC-h which gives a natural extension by lax modality of the well-known Gödel/Dummett logic LC of (finite) linear Kripke models. Our main results are that for finite linear intensional models L the intensional theory $iTh(L) = \{ p : \varphi \mid L \models p : \varphi \}$ characterises L and that iLC-h generates complete information about $iTh(L)$.

Our paper thus shows that the quantitative intensional information contained in the effort measure T can be abstracted away by the use of $\bigcirc$ and completely recovered by a suitable semantic interpretation of proofs.

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1 Motivation

The goal of this paper is to develop some of what we tentatively call "intensional model theory" for intuitionistic logic (IL), *i.e.* a model theory which does not only consider the valid or provable formulas of a logic but also the way in which these have been established.

Kripke semantics are probably the most elementary and convenient among the many ways of characterising pure validity in IL. Kripke models in themselves can be seen as an *intensional* refinement of the idea of a classical model of logic in which truth is relativised to "possible worlds" thought of as representing "construction stages" along which truth is established in a monotone and incremental fashion. Typically, in canonical models of IL worlds are built from intuitionistic theories (deductively closed subsets of formulas) embodying the knowledge available at a given stage of construction.

However, there is a distinct mismatch between the standard Kripke notion of validity at a world and the intensional idea of constructively established knowledge in the associated theory. If $\mathcal{T}$ is a theory in IL acting as a world in some (*e.g.* canonical) Kripke model then validity $\mathcal{T} \models \varphi \vee \psi$ forces the disjunction to be decided "on the spot", *i.e.* we must have $\mathcal{T} \models \varphi$ or $\mathcal{T} \models \psi$. On the other hand, deductive inference $\vdash$, which is our sole means of establishing constructive knowledge, does not generally satisfy the equivalence $\mathcal{T} \vdash \varphi \vee \psi$ iff $\mathcal{T} \vdash \varphi$ or $\mathcal{T} \vdash \psi$. This so-called *disjunction property* is a definitive feature of *constructive* theories $\mathcal{T}$ only. As a consequence the idea of defining validity in terms of constructive deducibility breaks down at this point. To bridge this gap the classical constructions of counter models must artifically close off theories under the disjunction property. This is done, typically, by a process of saturation in which all disjunctions such that $\mathcal{T} \vdash \varphi \vee \psi$ are systematically enumerated and the theory $\mathcal{T}$ extended by φ or ψ, depending on consistency. This decision, namely whether $\mathcal{T} \cup \{\varphi\}$ or $\mathcal{T} \cup \{\psi\}$ is consistent, is non-constructive in general and requires a classical meta-theory (act of speculation). A similar remark, of course, applies to existential quantification.

A well-known solution to this problem is to replace the standard interpretation of Kripke by Beth's relaxed notion of validity in which $\mathcal{T} \models \varphi \vee \psi$ iff there exists a bar β of theories extending $\mathcal{T}$ such that for all $\mathcal{T}' \in \beta$, $\mathcal{T}' \models \varphi$ or $\mathcal{T}' \models \varphi$, see *e.g.* [vD86]. Thus, $\mathcal{T} \models \varphi \vee \psi$ does not force the decision for φ or ψ to be made "on the spot" within $\mathcal{T}$ itself, but merely requires it to be unavoidable along all construction sequences (paths through the model frame) that may be taken from $\mathcal{T}$. In this way one then obtains a direct identification of intuitionistic theories with worlds in a canonical model such that $\mathcal{T} \models \varphi$ iff $\mathcal{T} \vdash \varphi$ without need for speculative saturation. In this sense, Beth models are more constructive than Kripke models and also more *intensional*. In Beth's interpretation the individual construction steps represented in the model correspond to merely mechanical calculations that can be abstracted from when we determine the validity of formulas in the model. So, the validity of $\varphi \vee \psi$ is relaxed to mean "up to mechanical calculations, φ or ψ is established". In contrast, in Kripke's interpretation each construction step (transition between worlds), in general, must be viewed as a genuine mental act that may be reflected in the validity of formulas. It is this *additional* level of intensionality of the Beth semantics that we are interested in here. In this vein, we use the term "intensional" semantics generically for variations of Beth semantics as refinements of

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the standard Kripke interpretation.

Beth models have been introduced originally to obtain constructive completeness proofs for intuitionistic logic [LEV74, Fri75, dS76, Vel76, Dum77, TvD88] and later shown to provide also an adequate Kripke-style model-theory (truth-value semantics) for realisability interpretations [TvD88, Lip91, MM87], which further underlines their intrinsic constructive nature. More work in this direction is cited in [Lip91]. Several generalisations of Beth's interpretation have emerged independently in a topological setting, specifically topos theory and the categorical analysis of logic. As discussed, *e.g.* in [Gol86], these reinterpret Grothendieck topologies as formalising a notion of "local truth" (Lavwere) which generalises Beth's relaxed interpretation of disjunction and existential quantification. The most well-known version is based on the notion of *covering systems* [Gol86]. A slightly more general formulation which does not require lattice properties is the following: A *covering system* on a frame $(W, \sqsubseteq)$ is an assignment of subsets $\mathrm{Cov}(w) \subseteq 2^W$ of worlds to every $w \in W$ satisfying the following three coherence axioms: (*i*) $\{w\} \in \mathrm{Cov}(w)$; (*ii*) if $U \in \mathrm{Cov}(w)$ and for every $x \in U$, $V_x \in \mathrm{Cov}(x)$, then $\bigcup_{x \in U} V_x \in \mathrm{Cov}(w)$; (*iii*) if $U \in \mathrm{Cov}(w)$ and $w \sqsubseteq v$, then $\{ x \mid \exists u \in U.\, u \sqsubseteq x \;\&\; v \sqsubseteq x \} \in \mathrm{Cov}(v)$. A covering system defines a modal operator

$$w \models \bigcirc\varphi \quad \text{iff} \quad \exists U \in \mathrm{Cov}(w).\, \forall u \in U.\, u \models \varphi.$$

Note that as a special case we may have $\emptyset \in \mathrm{Cov}(w)$ which means that w is a fallible world satisfying $\bigcirc$*false*. If $\models \varphi$ is validity in the standard Kripke sense, then $\models \bigcirc\varphi$ expresses validity of φ in a generalised Beth sense. For if we let $\mathrm{Cov}(w)$ be the set of bars for w, we obtain a covering system and $\bigcirc\varphi$ yields validity according to the Beth interpretation. The modal operator is characterised by the axioms

$$\begin{array}{lcl} \bigcirc I & : & \varphi \supset \bigcirc\varphi \\ \bigcirc M & : & \bigcirc\bigcirc\varphi \supset \bigcirc\varphi \\ \bigcirc S & : & (\bigcirc\varphi \,\&\, \bigcirc\psi) \supset \bigcirc(\varphi \,\&\, \psi) \end{array}$$

and the rule $\bigcirc F$: $\vdash \varphi \supset \psi \Rightarrow \bigcirc\varphi \supset \bigcirc\psi$. The associated logic we call (propositional) *lax logic* [FM97]. Modal operators $\bigcirc$ for local truth can be generated in various ways. For instance, in [Dra88] a suitable generalisation of path systems, the dual to covering systems, is used. Neighbourhood systems and topological congruences have been introduced in [Gol81] as a semantics for $\bigcirc$. In all of these cases $\bigcirc$ can be understood as a formal operator to abstract from some intensional model structure of a standard Kripke model. To obtain the generalised Beth interpretation, then, one simply decorates all atomic sentences, disjunctions, and existentials with the lax modality.

The "blurred" interpretation of formulas and the redundant intensional structure abstracted from by $\bigcirc$ is best exemplified by the (well-known) canonical (generalised) Beth model whose worlds are single closed formulas and accessibility $\varphi \sqsubseteq \psi$ defined as provable implication $\vdash \psi \supset \varphi$. One defines as covers $U \in \mathrm{Cov}(\varphi)$ those sets of formulas U that satisfy $\forall\theta \in U.\, \varphi \sqsubseteq \theta$ and $\forall\xi.\, (\forall\theta \in U.\, \xi \sqsubseteq \theta) \Rightarrow \xi \sqsubseteq \varphi$. If one then takes forcing of atomic sentences $\varphi \models \alpha$ to be provable equivalence $\vdash \varphi \supset \alpha$ and relaxes the interpretation of disjunction in the style of Beth

$$\varphi \models \bigcirc(\psi_1 \vee \psi_2) \quad \text{iff} \quad \exists U \in \mathrm{Cov}(\varphi).\, \forall\theta \in U.\, \theta \models \psi_1 \text{ or } \theta \models \psi_2$$

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then one proves for *all* propositions in which $\vee$ is guarded by $\bigcirc$ that $\varphi \models \psi \Leftrightarrow \ \vdash \varphi \supset \psi$. The beauty of this technique is that it establishes an extremely tight connection between validity and provability, and, as demonstrated in the works cited above, can be used to obtain surprisingly direct and constructive completeness proofs.

However, and this is the point of departure for this paper, the resulting canonical models are typically rather intensional (or even syntactic). The example discussed in the previous paragraph, which is a simplified version of that presented in [Lip91] to capture realisability, consists of lots of worlds with the same extension, *i.e.* worlds validating the same theory, but which are distinguished in the model simply because they are presented as syntactically different formulas. The modal operator $\bigcirc$, in the literature mostly left implicit in the semantic clauses for $\vee$, achieves the necessary abstraction from this intensional model structure. As this "redundant" intensional structure is generated by a formal proof system it appears natural to expect that in a truly constructive model this intensional structure could be linked with explicit proofs in the calculus. To be somewhat more precise let us assume, as in our example, that

$$\varphi \models \bigcirc\psi \quad \text{iff} \quad \exists U \in \mathrm{Cov}(\varphi).\ \forall \theta \in U.\ \theta \models \psi$$

defines extensional validity local to φ. Suppose further, as in the example, that $\varphi \models \psi$ is constructively equivalent to the existence of a constructive proof of $\varphi \supset \psi$. Then, in a calculus in which $\bigcirc$ is a first-class operator, one should have a way of relating actual proofs $\vdash p : \mathit{true} \supset \bigcirc\psi$ to the description of a cover U^* such that (provably) $\forall \theta \in U^*.\ \theta \supset \psi$. In this way, proofs would contain explicit information about the intensional meaning of ψ, *i.e.* the offset ("Beth-slack") between the worlds in which $\models \bigcirc\psi$ and those in which $\models \psi$ proper. Such a development, if it exists, would establish a stronger and more intensional connection between Kripke-Beth-style and realisability semantics compared to existing literature, where in the translation from realisability to Kripke-Beth models proof information is lost.

Our previous work on Lax Logic [Men93, FMW97, FMC01] provides some evidence that such a programme may be feasible, at least for special cases. What it shows, roughly speaking, is that $\bigcirc\psi$ can be viewed as stating "φ up to constraints" and that a proof of $\bigcirc\psi$ yields a description of a constraint γ such that $\gamma \supset \psi$ is provable. This previous work was motivated by applications, while the aim of this paper is to investigate this programme from a general logical perspective.

2 Contribution of our paper

Our exposition of Propositional Lax Logic PLL [FM97] and its model theory provides the background to this paper. From a model theoretic perspective, PLL arises from the elaboration of a standard Kripke model $M = (W, \sqsubseteq, V)$ into a Kripke *constraint* model $C = (W, \sqsubseteq, \sqsubseteq_m, V)$ by providing a binary classification of effort on the accessibility relation $\sqsubseteq$. This is just a subrelation $\sqsubseteq_m$ of $\sqsubseteq$. If $w \sqsubseteq_m u$ we may think of this as saying that u can be reached from w within bounded effort, while $w \sqsubseteq u$ merely expresses that u is reachable from w. We use the lax modal operator $\bigcirc$ to express properties of $\sqsubseteq_m$, with $\bigcirc\varphi$ expressing the fact that φ can always be satisfied after an $\sqsubseteq_m$ step, *i.e.* within bounded effort. $\sqsubseteq_m$ induces a covering

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system on $(W, \sqsubseteq)$ such that $U \in \mathrm{Cov}(w)$ if and only if for all $v \sqsupseteq w$, there exist $u \in U$ and $u' \sqsupseteq u$ such that $v \sqsubseteq_m u'$. This construction gives us an elementary class of generalised Beth models which are especially convenient as carrier structures for intensional information. Our plan is to provide a refined interpretation of $\bigcirc\varphi$ as meaning "φ can be satisfied up to bounded effort", where proof objects $p : \bigcirc\varphi$ give a concrete measure of the amount of effort involved. The effort measure is a decoration of the pairs related by $\sqsubseteq_m$ that leads to a definition of intensional model in section 3. Rather than attempting to give a general theory for this idea, we concentrate on a particular case study, namely an intensional refinement iLC of the much-studied Gödel/Dummett logic LC [Dum59]. In this special case, we are able to give an intensional strengthening of existing soundness and completeness results by demonstrating an exact correspondence between proofs $\cdot \vdash p : \varphi$ and fine structure $\cdot \models p : \varphi$. This is achieved through a natural extension of Dummett's axiomatisation of LC. Our intensional semantics is a variant of Medvedev's logic of finite problems [Med66], a logic studied in depth by P. Miglioli and others in [MMO+89].

Main results of the paper In the first part of this paper we define the notion of intensional Kripke model as a structure $M = (W, \sqsubseteq, V, T)$ where $T(w, u)$ is a function giving an effort measure in $\mathbb{N} \cup \{\infty\}$ for each $\sqsubseteq$-related pair (w, u). We show that $\bigcirc$ embodies the abstraction involved in passing from "φ true up to bounded effort" to "φ true outright" and in Theorem 3.4 we capture this abstraction process in terms of both models and theories.

In the second part we extend our presentation of plain validity $M \models \varphi$ by a refined notion of **intensional validity** $M \models p : \varphi$ and introduce a Hilbert-style calculus iLC-h for intensional validity on finite linear models L. In Theorem 5.1 we show that the intensional theory $iTh(L) = \{ p : \varphi \mid L \models p : \varphi \}$ characterises L and in Theorem 7.4 that iLC-h generates complete information about $iTh(L)$.

3 Intensional Kripke Models and plain semantics

As usual, an (intuitionistic) Kripke model is a triple $(W, \sqsubseteq, V)$ where $\sqsubseteq$ is a partial order on W and V is a hereditary valuation, *i.e.* a monotone map from W to sets of propositional constants.

Definition 3.1 *An **intensional** Kripke model is an (intuitionistic) Kripke model $M = (W, \sqsubseteq, V)$ together with a directed **effort measure** $T : \sqsubseteq \to \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ satisfying the following **effort laws** whenever $u \sqsubseteq v \sqsubseteq w$:*

1. $T(w, w) = 0$
2. $T(u, w) \leq T(u, v) + T(v, w)$
3. $max\,(T(u, v), T(v, w)) \leq T(u, w)$

We say M is finite *if W is finite and for each $w \in W$, $V(w)$ is a finite set of atoms.*

$T(u, w)$ measures "worst case effort" between u and w. As an example, this could correspond to a maximal separation time between u and w. Clearly this effort should be zero if u and

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w are the same state; condition 2 is the Triangle Law familiar from metric spaces and we call condition 3 the Entropy Law because it captures the irreversible nature of the effort expended in the step from u to v under $\sqsubseteq$. The effort is irreversible in the sense that if it costs us e units of effort to go from u to v then we cannot go from u to any point beyond v without expending at least e units of effort. If we wish to think of u and w as stages in the construction of a mathematical universe, then $T(u, w) < \infty$ corresponds to "bounded effort" which we may picture as a mechanical construction step and $T(u, w) = \infty$ corresponds to "unbounded effort" which we may picture as a creative step.

An example of an intensional Kripke model L based on a linear order $\sqsubseteq$ is given in Fig. 1. Solid arrows such as that between w_2 and w_3 correspond to bounded effort and dotted arrows such as that between w_3 and w_4 correspond to unbounded effort. Regions of bounded effort are shaded in grey and T is indicated by labels on solid arrows within the region containing w_5.

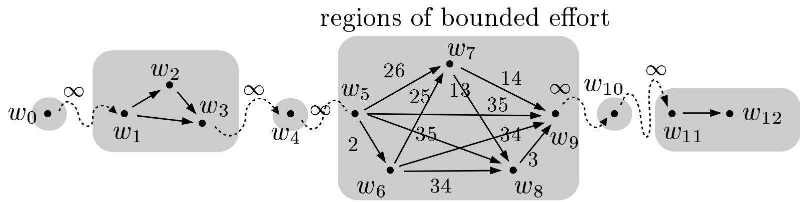

Figure 1: A linear intensional Kripke model L

Definition 3.2 *M is **extensional** if $T(w, v) < \infty \Rightarrow w = v$ (coarsest possible view of effort).*

Extensional models arise as **abstractions** of intensional models. According to the extensional viewpoint, everything that can be known with bounded effort might just as well be known outright. To formalise this abstraction we introduce a **modal operator** $\bigcirc\varphi$ which means "φ within bounded effort". Consider the intuitionistic language defined by

$$\varphi \quad ::= \quad \alpha \mid \mathit{true} \mid \mathit{false} \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \,\&\, \varphi_2 \mid \varphi_1 \supset \varphi_2 \mid \bigcirc\varphi$$

where α is any propositional atom. As usual we define $\neg\varphi$ to be $\varphi \supset \mathit{false}$.

Definition 3.3 (Plain semantics) *φ is **validated at world** $w \in W$ of the intensional model $M = (W, \sqsubseteq, V, T)$ written $M, w \models \varphi$, according to the standard Kripke semantics [Kri63] for atoms, conjunction, disjunction and implication, e.g. $M, w \models \varphi_1 \supset \varphi_2$ iff $\forall u \sqsupseteq w.\, M, u \models \varphi_1 \Rightarrow M, u \models \varphi_2$. We extend this semantics to account for the modality $\bigcirc$ by*

$$M, w \models \bigcirc\varphi \quad \mathit{iff} \quad \forall v.\, w \sqsubseteq v \Rightarrow \exists u.\, v \sqsubseteq u \;\&\; T(v, u) < \infty \;\&\; u \models \varphi.$$

*Then M **validates** φ, $M \models \varphi$, iff $\forall w \in W.\, M, w \models \varphi$ and φ is **valid**, $\models \varphi$, iff $\forall M.\, M \models \varphi$. We abuse this notation slightly by writing $M, U \models \varphi$ if $M, u \models \varphi$ for every $u \in U \subseteq W$.*

Note that the clause for $\bigcirc$ ensures that truth remains hereditary when we add $\bigcirc$, *i.e.* $M, w \models \varphi$ and $w \sqsubseteq v$ implies $M, v \models \varphi$. We call this the plain semantics because it only assigns meanings

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to formulas unadorned with proof objects. As discussed earlier, it can also be cast in terms of covering systems. The relationship between our Kripke semantics for $\bigcirc$ and Goldblatt's neighbourhood models [Gol81] is discussed further in [FM97].

Let $M = (W, \sqsubseteq, V, T)$ be a *finite* intensional Kripke model. The **extensional abstraction** $\mathsf{Ext}(M)$ is M restricted to the T**-stable** worlds of M, where a world w is T-stable if for all $v \in W$, if $T(w, v) < \infty$ then $v = w$, *i.e.* there are no new worlds accessible from w in bounded effort. More formally, $\mathsf{Ext}(M) = (J, \sqsubseteq', V', T')$ where J is the set of T-stable worlds of W, $\sqsubseteq'$ is the restriction of $\sqsubseteq$ to $J \times J$ *i.e.* $\sqsubseteq' = \sqsubseteq \cap J \times J$, V' is the restriction of V to J and T' is the restriction of T to $\sqsubseteq'$. Fig. 2 gives an example of the abstraction process as applied to the intensional model L of Fig. 1. The abstraction $M \mapsto \mathsf{Ext}(M)$ can be captured by

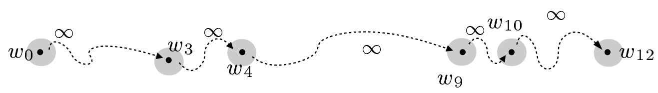

Figure 2: Abstracted intensional model $\mathsf{Ext}(L)$ of Fig. 1

formal theory, too. Let ζ be a $\bigcirc$-free proposition. Let $K(\zeta)$ be the result of replacing every subformula $\zeta_1 \supset \zeta_2$ in ζ by the **Kleisli implication** $\zeta_1 \supset \bigcirc\zeta_2$ and let $\mathsf{Kleisli}(\zeta)$ be $\bigcirc K(\zeta)$. For example if $\zeta = \alpha \,\&\, (\beta \supset \gamma \supset \delta)$ then $\mathsf{Kleisli}(\zeta) = \bigcirc(\alpha \,\&\, (\beta \supset \bigcirc(\gamma \supset \bigcirc\delta)))$. We use the term "Kleisli" by analogy with the Kleisli category of a monad—see for example [Lan88].

Theorem 3.4 *Let M be a finite intensional model. Then* $\mathsf{Ext}(M) \models \zeta \Leftrightarrow M \models \mathsf{Kleisli}(\zeta)$.

Proof: Let $M = (W, \sqsubseteq, V, T)$ be a *finite* intensional model. For any $w \in W$ let $w\!\sqsubseteq$ denote $\{u \mid w \sqsubseteq u\}$, the set of worlds accessible from w. Let $J \subseteq W$ denote the T-stable elements of W and consider the set $J_w = (w\!\sqsubseteq) \cap J$. Now let $\varepsilon(w) = \{v \in J_w \mid \forall u.\, u \sqsubseteq v \,\&\, u \in J_w \Rightarrow u = v\}$. That is to say, $\varepsilon(w)$ is the set of all minimal T-stable elements on $\sqsubseteq$-paths in W beginning at w. Because W is finite, every $\sqsubseteq$-path from w is finite and therefore contains a member of J_w, so contains a least such member. Thus $\varepsilon(w)$ contains at least one element. $\varepsilon(w)$ is constructed in such a way that if $w \sqsubseteq u \sqsubseteq v$ and $v \in \varepsilon(w)$ then all three efforts $T(w, u)$, $T(u, v)$ and $T(w, v)$ are finite. It is easy to show by induction on $\bigcirc$-free formulas ζ that $\mathsf{Ext}(M), \varepsilon(w) \models \zeta \Leftrightarrow M, \varepsilon(w) \models K(\zeta)$. Now we use the observation that for any finite model M and formula φ, $M, \varepsilon(w) \models \varphi$ *iff* $M, w \models \bigcirc\varphi$ to deduce that for arbitrary $w \in W$ $\mathsf{Ext}(M), \varepsilon(w) \models \zeta \Leftrightarrow M, w \models \bigcirc K(\zeta) = \mathsf{Kleisli}(\zeta)$ and hence, since the set J of worlds of $\mathsf{Ext}(M)$ is of the form $\bigcup_{w \in W} \varepsilon(w)$, we have $\mathsf{Ext}(M) \models \zeta \Leftrightarrow M \models \mathsf{Kleisli}(\zeta)$. ∎

We have fulfilled our promise to show that $\bigcirc$ provides an abstraction mechanism, for we can recover the standard intuitionistic theory of an extensional model from the Kleisli theory of any of its intensional refinements. In this sense, we can see that adding $\bigcirc$ and intensional structure gives a conservative extension of intuitionistic logic. We can however do more, and this is the main message of this paper: The intensional information can also be recovered from an intensional theory, if we take proof objects into account. The next step in this direction is thus to find an axiomatisation for $\bigcirc$. It is easy to see that the axioms $\bigcirc I : \varphi \supset \bigcirc\varphi$ and $\bigcirc M : \bigcirc\bigcirc\varphi \supset \bigcirc\varphi$ and the rule $\bigcirc F :$ from $\varphi \supset \psi$ infer $\bigcirc\psi \supset \bigcirc\varphi$ are sound with respect to the plain semantics for intensional models, and the axiom $\mathit{false}_\circ : \bigcirc\mathit{false} \supset \mathit{false}$ follows

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from the fact that the intensional models of this paper have no fallible worlds. These are the characteristic properties of a **closure operator**. A further axiom $\bigcirc S : (\bigcirc\varphi \& \bigcirc\psi) \supset \bigcirc(\varphi \& \psi)$ ensures that $\bigcirc$ is **strongly extensional**, *i.e.* $\models (\varphi \equiv \psi) \supset (\bigcirc\varphi \equiv \bigcirc\psi)$.

If we add a complete set of axioms and rules for IPC we obtain a calculus $\mathsf{SLL} = \mathsf{IPC} + \bigcirc I + \bigcirc M + \bigcirc F + \mathit{false}_\circ + \bigcirc S$ that we call **Strong Lax Logic**, where the term "strong" indicates the presence of the axiom $\mathit{false}_\circ$. The fact that this calculus is complete for finite intensional Kripke models is essentially the content of Theorem 4.5 of [FM97]:

Theorem 3.5 $\mathsf{SLL} \vdash \varphi$ *iff for all finite intensional models* M, $M \models \varphi$.

However, SLL is not *intensionally* adequate, because the SLL-theory $Th(M) = \{\, \varphi \mid M \models \varphi \,\}$ of $M = (W, \sqsubseteq, V, T)$ characterises $(W, \sqsubseteq, V)$ up to a surjective p-morphism (bisimulation equivalence) and also tells us if $T(u, v) < \infty$ or $T(u, v) = \infty$ but it does not tell us anything about the **absolute value** of $T(u, v)$. We are missing the quantitative information on effort measures which will be accounted for in the next section, in which we provide a precise computational interpretation for the terms $\bigcirc I$, $\bigcirc M$, $\bigcirc S$, $\bigcirc F$ and $\mathit{false}_\circ$.

4 Intensional semantics for linear models

The remainder of this paper focuses on a simple class of intensional models $L = (W, \sqsubseteq, V, T)$ where W and V are finite, $(W, \sqsubseteq)$ is a linear ordering and V is strictly increasing: $w \sqsubset v$ implies $V(w) \subsetneq V(v)$ (*i.e.* the model L is finite, linear and irredundant). We call these **enriched sequence models** (*esms*). Fig. 1 showed a typical example. Note that disjunction $\vee$ is redundant over linear models [Dum59] since it can be defined by the construction $\varphi \oplus \psi =_{df} (\varphi \supset \psi) \supset \psi \;\&\; (\psi \supset \varphi) \supset \varphi$ for which $L \models \varphi \oplus \psi$ *iff* $L \models \varphi$ or $L \models \psi$. We therefore restrict our logical language to $\vee$-free formulas from now on. Theorem 3.5 provides a motivation for our focus on finite models, and we shall see that these are also sufficient in the linear case.

Plain formulas φ will be interpreted as domains $[\![\varphi]\!]$ of hereditarily monotone effort bounds. Given a refined formula $p : \varphi$, p will be interpreted as an element of $[\![\varphi]\!]$ and the formula $p : \varphi$ as a statement about the role played by p in an intensional model. More precisely, $[\![\varphi]\!]$ will correspond to a hereditarily monotone subdomain of a type $[\varphi]$ within FT, where FT is the set of finite type functionals generated from the singleton type $\underline{1} = \{0\}$ and $\mathbb{N}$ using the operations of Cartesian product $\times$ and full function space $\to$.

Definition 4.1 *Each type* $t \subseteq \mathsf{FT}$ *carries a partial order* $\preccurlyeq_t$ *which is lifted pointwise from the natural order* $\leq$ *on* $\mathbb{N}$*. It also has a minimum element* 0_t *and binary functions* max_t, min_t*. We may define these objects by recursion on the structure of* t*:*

1. $\preccurlyeq_{\underline{1}} =_{df} \{(0,0)\}$, $0_{\underline{1}} =_{df} 0$ *and* $max_{\underline{1}}(0,0) = min_{\underline{1}}(0,0) =_{df} 0$

2. $\preccurlyeq_{\mathbb{N}} =_{df} \leq$, $0_{\mathbb{N}} =_{df} 0$ *and* $max_{\mathbb{N}}$, $min_{\mathbb{N}}$ *are the standard maximum and minimum functions on* $\mathbb{N}$

3. $\preccurlyeq_{s\times t} =_{df} \{((p_1, q_1), (p_2, q_2)) \mid p_1 \preccurlyeq_s p_2 \;\&\; q_1 \preccurlyeq_t q_2\}$, $0_{s\times t} =_{df} (0_s, 0_t)$, $max_{s\times t}((p_1, q_1), (p_2, q_2)) =_{df} (max_s(p_1, p_2), max_t(q_1, q_2))$ *and* $min_{s\times t}((p_1, q_1), (p_2, q_2)) =_{df} (min_s(p_1, p_2), min_t(q_1, q_2))$

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4. $\preccurlyeq_{s\to t} =_{df} \{(p,q) \mid \forall r \in s.\, p\, r \preccurlyeq q\, r\}$, $0_{s\to t} =_{df} \lambda x.0_t$, $max_{s\to t}(p,q) =_{df} \lambda x.\, max_t(p\, x, q\, x)$ *and* $min_{s\to t}(p,q) =_{df} \lambda x.\, min_t(p\, x, q\, x)$

The absence of a sum type to represent disjunction means that each type $t \subseteq \mathsf{FT}$ carries a lattice structure $(t, \preccurlyeq_t, max_t, min_t, 0_t)$ which has greatest lower bound $min_t\, S$ for each inhabited subset S of t, so that $0_t = min_t\, t$ is the least element under $\preccurlyeq_t$ and every subset S bounded above by p has least upper bound $max_t\, S \preccurlyeq_t p$. We use $\preccurlyeq_t$, max_t, min_t and 0_t as polymorphic constants by suppressing the suffix t wherever we can—for example we may write $(0,0) = 0$ instead of $(0_s, 0_t) = 0_{s\times t}$. We will use the ordering $\preccurlyeq$ to compare effort bounds. If $p \preccurlyeq q$ then p is a better or tighter bound than q, so that q will inherit the properties of p, *i.e.* $L, w \models p : \varphi \;\&\; p \preccurlyeq q \Rightarrow L, w \models q : \varphi$.

It is technically convenient to restrict our attention to what we term the hereditarily monotone functionals of FT. If $p \in t \to s$ then p is by definition monotone if $\forall r, q.\, q \preccurlyeq r \Rightarrow p\, q \preccurlyeq p\, r$. Then max and min turn out to be not only monotone but hereditarily monotone as defined by the following constructions.

Definition 4.2 *For every φ the sets $[\varphi] \subset \mathsf{FT}$ of **potential effort bounds** and $[\![\varphi]\!] \subseteq [\varphi]$ of **proper effort bounds** for φ are given as follows: $[true]$, $[\![true]\!]$, $[false]$, $[\![false]\!]$, $[\alpha]$ and $[\![\alpha]\!]$ are all defined to be $\underline{1}$. $[\varphi \,\&\, \psi] =_{df} [\varphi] \times [\psi]$ and $[\![\varphi \,\&\, \psi]\!] =_{df} [\![\varphi]\!] \times [\![\psi]\!]$. $[\varphi \supset \psi] =_{df} [\varphi] \to [\psi]$ and $[\![\varphi \supset \psi]\!] =_{df} \{f \in [\varphi] \to [\psi] \mid f$ is monotone $\&\; \forall p \in [\![\varphi]\!].\, f\, p \in [\![\psi]\!]\}$. Finally $[\bigcirc\varphi] =_{df} \mathbb{N} \times [\varphi]$ and $[\![\bigcirc\varphi]\!] =_{df} \mathbb{N} \times [\![\varphi]\!]$.*

We write $p \preccurlyeq q : \varphi$ when $\{p, q\} \subseteq [\![\varphi]\!]$ and $p \preccurlyeq_{[\varphi]} q$. We may now define the hereditarily monotone functionals of FT as those functionals in $[\![\varphi]\!]$ for some φ.

Lemma 4.3 *We omit the proof of the following two facts. Firstly, if ζ is $\bigcirc$-free then $[\![\zeta]\!] = \{0\}$, that is, $\bigcirc$-free formulas carry no intensional information. We call such a ζ a **unit** formula. In this case, $L, w \models 0 : \zeta$ precisely when ζ is true at w in L in the standard sense as a formula of LC on linear models. Secondly, for every φ the structure $([\![\varphi]\!], \preccurlyeq_{[\![\varphi]\!]}, 0_{[\![\varphi]\!]}, max_{[\![\varphi]\!]}, min_{[\![\varphi]\!]})$ is a lattice with greatest lower bound $min\, S$ for any inhabited $S \subseteq [\![\varphi]\!]$.*

We now define the intensional semantics for pairs $p : \varphi$.

Definition 4.4 (Intensional semantics) *φ is **validated at world** $w \in W$ of the enriched sequence model $L = (W, \sqsubseteq, V, T)$ **with bound** $p \in [\![\varphi]\!]$, written $L, w \models p : \varphi$, according to the clauses*

$$
\begin{array}{lcl}
L, w & \models & 0 : true \\
L, w & \models & 0 : \alpha \quad \textit{iff} \quad \alpha \in V(w) \\
L, w & \models & (p_1, p_2) : \varphi_1 \,\&\, \varphi_2 \quad \textit{iff} \quad L, w \models p_1 : \varphi_1 \;\;\&\;\; L, w \models p_2 : \varphi_2 \\
L, w & \models & p : \varphi \supset \psi \quad \textit{iff} \quad \forall q \in [\![\varphi]\!].\, \forall u \sqsupseteq w.\, L, u \models q : \varphi \Rightarrow L, u \models p\, q : \psi \\
L, w & \models & (\delta, p) : \bigcirc\varphi \quad \textit{iff} \quad \forall v \sqsupseteq w.\, \exists u \sqsupseteq v.\, T(v, u) \leq \delta \;\&\; L, u \models p : \varphi
\end{array}
$$

*L **validates** φ **with bound** p, written $L \models p : \varphi$, iff $\forall w \in W.\, L, w \models p : \varphi$ and φ is **valid with bound** p, written $\models p : \varphi$ iff $\forall L.\, L \models p : \varphi$. In this case we call p a uniform bound for φ.*

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This gives us set-theoretic realisability interpretation in the style of Kolmogoroff and Medvedev [Kol32, Med66], though with some significant differences which we discuss further in section 8. Note that this semantics refines the plain semantics of Definition 3.3 in a strong sense:

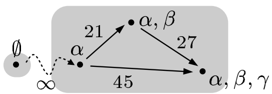
Lemma 4.5 $L \models \varphi \;\Leftrightarrow\; \exists p \in [\![\varphi]\!].\, L \models p : \varphi$.

Note also that, due to the linearity of L, we may simplify the clause for $\bigcirc$ to $L, w \models (\delta, p) : \bigcirc\varphi$ *iff* $\exists u \sqsupseteq w.\, T(w,u) \leq \delta \;\&\; L, u \models p : \varphi$.

Lemma 4.6 *The following properties of intensional validity can be established by induction on formulas* φ*. (1) validity is hereditary along* $\sqsubseteq$*:* $(L, w \models p : \varphi \;\&\; w \sqsubseteq u) \Rightarrow L, u \models p : \varphi$*. (2) validity with bounds is preserved along* $\preccurlyeq$*:* $(L, w \models p : \varphi \;\&\; p \preccurlyeq q : \varphi) \Rightarrow L, w \models q : \varphi$*. (3) if* L *extensional, then* $L \models \lambda(\delta, p).\, p : \bigcirc\varphi \supset \varphi$*.*

5 Intensional expressiveness

Enriched sequence models can be characterised already in a simple propositional fragment with restricted use of $\bigcirc$. To illustrate the key points of the construction we give a simple example. Our aim is to specify the following model L



in atoms $\{\alpha, \beta, \gamma, \eta\}$. This is achieved by the following intensional specification $IS = \{P_1, \ldots, P_6\}$ where we write *e.g.* $21 : \alpha \supset \bigcirc\gamma$ instead of the more accurate $\lambda x.(21, 0) : \alpha \supset \bigcirc\gamma$.

P_1	$=_{df}$	$0 : \neg\neg(\alpha \,\&\, \beta \,\&\, \gamma \,\&\, \neg\eta)$	*the final valuation has* α, β, γ *and not* η
P_2	$=_{df}$	$0 : (\beta \supset \alpha) \,\&\, ((\alpha \supset \beta) \supset \beta)$	α *comes strictly before* β
P_3	$=_{df}$	$0 : (\gamma \supset \beta) \,\&\, ((\beta \supset \gamma) \supset \gamma)$	β *comes strictly before* γ
P_4	$=_{df}$	$21 : \alpha \supset \bigcirc\beta$	*there are at most* 21 *effort units from* α *to* β
P_5	$=_{df}$	$45 : \alpha \supset \bigcirc\gamma$	*there are at most* 45 *effort units from* α *to* γ
P_6	$=_{df}$	$27 : \beta \supset \bigcirc\gamma$	*there are at most* 27 *effort units from* β *to* γ

The first three components P_1, P_2, P_3 specify the linear structure of L as a finite linear Kripke model (*flm*) $(W, \sqsubseteq, V)$, and given this structure, the last three components P_4, P_5, P_6 specify the effort measure on $\sqsubseteq$. These intensional facts specify all there is to know about L, as any other model satisfying IS is a faster suffix of L, in a sense we define below.

Theorem 5.1 (Intensional expressiveness)
For every esm L *let* $iTh(L) =_{df} \{p : \varphi \mid L \models p : \varphi\}$*. Then* $L_1 = L_2 \Leftrightarrow iTh(L_1) = iTh(L_2)$*.*

Proof: We sketch a proof of this result because it illustrates the key features of our proof of intensional completeness. First we summarise what are essentially the main constructions in Dummett's completeness proof for LC [Dum59].

Let $\mathbb{A}$ be a finite set of atoms and set $\mathbb{A}^+ = \mathbb{A} \cup \{true, false\}$. We call any proposition of the form $\alpha \supset \beta$ and $(\alpha \supset \beta) \supset \beta$ where $\alpha, \beta \in \mathbb{A}^+$ an *ordering proposition* in

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$\mathbb{A}$. These are useful to specify the relative ordering in which the atoms $\mathbb{A}$ are "switched on" in a linear model, interpreting *true* and *false* as the fictive beginning and end of the model, respectively. Up to provable equivalence (in IPC) ordering propositions are the forms *true*, *false*, α, $\neg\alpha$, $\neg\neg\alpha$, $(\alpha \supset \beta)$, $(\alpha \supset \beta) \supset \beta$, where $\alpha, \beta \in \mathbb{A}$. Now let $L = (W, \sqsubseteq, V)$ be an *flm*. We say that L has *valuation in* $\mathbb{A}$, or *signature* $\mathbb{A}$ if $V(w) \subseteq \mathbb{A}$ for all $w \in W$. If L is irredundant we may identify a world w with the finite set $V(w) \subseteq \mathbb{A}$ and the conjunction $\bigwedge V(w) =_{df} \bigwedge_{\alpha \in V(w)} \alpha$ and call it a (proper) *state* of L. Sometimes we also consider *true* and *false* as *generalised* states, *viz.* the beginning and end of the model. Another *flm* $L' = (W', \sqsubseteq', V')$ is called a *suffix* of L, written $L' \precsim L$, if there exists a p-morphism from L' to L, *i.e.* a monotone mapping of worlds $p : W' \to W$ such that for all $\forall w' \in W'.\ V'(w') = V(p(w'))$ and $\forall w' \in W', v \in W.\ p(w') \sqsubseteq v \Rightarrow \exists v' \in W'.\ w' \sqsubseteq' v'\ \&\ p(v') = v$. If $L' \precsim L$, then because of irredundancy, L' is (up to a trivial isomorphism) simply a final sub-model of L. Now the set of all ordering propositions that hold true of L completely capture the structure of L up to $\precsim$. If L is a *flm* in signature $\mathbb{A}$, we define its *characteristic* proposition to be $\chi_{\mathbb{A}}(L) =_{df} \bigwedge\{\, \kappa \mid \kappa$ ordering proposition in $\mathbb{A}^{+}\ \&\ L \models \kappa \,\}$. Note that $\chi_{\mathbb{A}}(L)$ is a unit formula, so can be satisfied by at most one proof object. One then shows that for any two *flm*s L and L' in signature $\mathbb{A}$, $L' \models \chi_{\mathbb{A}}(L)$ *iff* $L' \precsim L$. So if two irredundant *esm*s L and L' of signature $\mathbb{A}$ have the same intensional theory then in particular each satisfies the other's characteristic proposition and thus they have the same underlying *flm*.

To also deal with the effort measure of a *esm* L of signature $\mathbb{A}$ we define its *characteristic* proposition $\chi^{\circ}_{\mathbb{A}}(L)$ to be $\bigwedge\{0 : \chi_{\mathbb{A}}(L)\} \cup \{\, p : \sigma \supset \bigcirc\tau \mid \sigma \sqsubseteq \tau\ \&\ \nu(p) = T(\sigma, \tau) < \infty \,\}$, where p corresponds to $T(\sigma, \tau)$ under the isomorphism $\nu : (\underline{1} \times \cdots \times \underline{1} \to \mathbb{N} \times \underline{1} \times \cdots \times \underline{1}) \cong \mathbb{N}$ and $\bigwedge\{p_1 : \varphi_1, \ldots, p_m : \varphi_m\} =_{df} (p_1, \ldots, p_m) : \varphi_1\ \&\ \cdots\ \&\ \varphi_m$. We extend the definition of suffix model to *esms*; if $L = (W, \sqsubseteq, V, T)$ and $L' = (W', \sqsubseteq', V', T')$ then $L' \precsim L$ *iff* $(W', \sqsubseteq', V')$ is a suffix of $(W, \sqsubseteq, V)$ such that $T'(w', v') \leq T(w', v')$ whenever $w' \sqsubseteq' v'$. If $L' \precsim L$ we say that L' is a **faster suffix** of L. Again we can show that for any two *esms* L and L' in signature $\mathbb{A}$, $L' \models \chi^{\circ}_{\mathbb{A}}(L)$ *iff* $L' \precsim L$, so that if *esms* L and L' have the same intensional theory then $L \precsim L'$ and $L' \precsim L$, which means that $L = L'$. ∎

We call a formula θ **elementary** if $\bigcirc$ does not occur on the left of any $\supset$ or inside any other $\bigcirc$. Elementary propositions include all $\bigcirc$-free propositions, and they always have $[\![\theta]\!] \cong \mathbb{N}^k$ for some $k \geq 0$, where $s \cong t$ represents a $\preccurlyeq$-preserving isomorphism between s and t. Note that any characteristic proposition $\chi^{\circ}(L)$ of an *esm* L is elementary, so we have shown that an enriched sequence model is actually characterised by its elementary theory.

6 The intensional calculus iLC-h

We now present a Hilbert-style calculus iLC-h which we shall eventually show to be intensionally sound and complete for *esm*s over elementary formulas. For the axiomatisation of iLC-h we take the intensional axiomatisation of IPC (essentially just the simply-typed λ-calculus) given in Fig. 3 plus the intensional axioms and rules of Fig. 4. This gives an intensional presentation of the calculus SLL. To this we add the following scheme

$$WLC =_{df} \lambda(p,\, q).\ max(p\,0,\ q\,0) : ((\zeta_1 \supset \zeta_2) \supset \varphi\ \&\ (\zeta_2 \supset \zeta_1) \supset \varphi) \supset \varphi$$

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$$
\begin{array}{rcl}
K & =_{df} & \lambda p,\, q.\, p \;:\; \varphi \supset (\psi \supset \varphi) \\
S & =_{df} & \lambda p,\, q,\, r.\, (p\; r)\; (q\; r) \;:\; (\varphi \supset (\psi \supset \chi)) \supset (\varphi \supset \psi) \supset (\varphi \supset \chi) \\
C & =_{df} & \lambda p,\, q.\, (p,\, q) \;:\; \varphi \supset \psi \supset (\varphi \,\&\, \psi) \\
\pi_0 & =_{df} & \lambda(p,\, q).\, p \;:\; (\varphi \,\&\, \psi) \supset \varphi \\
\pi_1 & =_{df} & \lambda(p,\, q).\, q \;:\; (\varphi \,\&\, \psi) \supset \psi \\
N & =_{df} & 0 \;:\; \mathit{false} \supset \varphi
\end{array}
$$

$$
\frac{\vdash p : \varphi \supset \psi \qquad \vdash q : \varphi}{\vdash p\, q : \psi}\, MP
$$

Figure 3: Intensional axiomatisation of IPC

$$
\begin{array}{rcl}
\bigcirc I & =_{df} & \lambda p.\, (0,\, p) \;:\; \varphi \supset \bigcirc\varphi \\
\bigcirc M & =_{df} & \lambda(\delta_1,\, (\delta_2,\, p)).\, (\delta_1 + \delta_2,\, p) \;:\; \bigcirc\bigcirc\varphi \supset \bigcirc\varphi \\
\bigcirc S & =_{df} & \lambda((\delta_1,\, p),\, (\delta_2,\, q)).\, (\mathit{max}\, (\delta_1,\, \delta_2),\, (p,\, q)) :\; (\bigcirc\varphi \,\&\, \bigcirc\psi) \supset \bigcirc(\varphi \,\&\, \psi) \\
\mathit{false}_\circ & =_{df} & 0 \;:\; \bigcirc\mathit{false} \supset \mathit{false}
\end{array}
$$

$$
\frac{\vdash p : \varphi \supset \psi}{\vdash \lambda(\delta, x).\, (\delta, p\, x) : \bigcirc\varphi \supset \bigcirc\psi}\, \bigcirc F
$$

Figure 4: Intensional axioms and rules for iLC-h

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for which φ may be arbitrary and ζ is restricted to $\bigcirc$-free propositions. Note that for extensional models ($\bigcirc\varphi \equiv \varphi$) iLC collapses to LC, which is complete for linear models [Dum59]. Proof theoretically, it can also be shown that iLC-h is a conservative extension of LC. The "W" in WLC stands for "weak", reflecting the restriction on ζ. This restriction is necessary, as we shall see below.

The following Deduction Theorem is crucial in that it allows us to use the more explicit and convenient λ-notation to denote proof objects in iLC-h rather than the unwieldy combinator language.

Theorem 6.1 (Deduction Theorem) *For every derivation* $\vec{x} : \Phi, y : \varphi_1 \vdash p : \varphi_2$ *in* iLC-h *there exists a derivation* $\vec{x} : \Phi \vdash \lambda y.\, p : \varphi_1 \supset \varphi_2$.

Proof: The Deduction Theorem holds for the Hilbert system of IPC. Any axioms that we add to it, such as $\bigcirc I$, $\bigcirc M$, $\bigcirc S$, and $false_\circ$ preserve this property. The only potential stumbling block for "λ-abstraction" is the new rule $\bigcirc F$. But this can be dealt with, too. The trick is to replace every application $\lambda y.\, \bigcirc F(p) : \varphi_1 \supset \bigcirc\varphi_{21} \supset \bigcirc\varphi_{22}$ for $\vec{x} : \Phi, y : \varphi_1 \vdash p : \varphi_{21} \supset \varphi_{22}$ of the Deduction Theorem by $curry\,(\bigcirc F(\lambda z.\ (\lambda y.\ p\,(\pi_2\, z))\,(\pi_1\, z)) \circ \bigcirc S \circ (\bigcirc I \times Id))$, where $z : \varphi_1 \,\&\, \varphi_{21}$, $\bigcirc S : (\bigcirc\varphi_1 \,\&\, \bigcirc\varphi_{21}) \supset \bigcirc(\varphi_1 \,\&\, \varphi_{21})$, $\bigcirc I : \varphi_1 \supset \bigcirc\varphi_1$, $Id : \bigcirc\varphi_{21} \supset \bigcirc\varphi_{21}$, and where $curry$ is the well-known currying combinator of type $((\varphi_1 \,\&\, \bigcirc\varphi_{21}) \supset \bigcirc\varphi_{22}) \supset \varphi_1 \supset \bigcirc\varphi_{21} \supset \bigcirc\varphi_{22}$, which can be constructed in IPC already. ∎

7 Soundness and completeness of iLC-h

Since decorated formulas $p : \varphi$ are first-class objects in our logic, it makes sense to define a semantic consequence relation by

$$p_1 : \varphi_1, p_2 : \varphi_2 \ldots, p_n : \varphi_n \quad \models \quad p : \psi \tag{1}$$

if for all esm L, $\forall i.\ L \models p_i : \varphi_i$ implies $L \models p : \psi$, or equivalently, if for all esm L and worlds w, $\forall i.\ L, w \models p_i : \varphi_i$ implies $L, w \models p : \psi$. This definition works whenever the p_i (p) are closed terms denoting elements of $[\![\varphi_i]\!]$ ($[\![\varphi]\!]$). On the formal side the Hilbert calculus iLC-h derives judgements of the form

$$x_1 : \varphi_1, x_2 : \varphi_2 \ldots, x_n : \varphi_n \quad \vdash \quad q : \psi \tag{2}$$

meaning that q is a proof of ψ from the assumptions $x_i : \varphi_i$, where the x_i represent arbitrary effort bounds $x_i \in [\![\varphi_i]\!]$. It is important to keep in mind that derivations are always parametric in the effort bounds of the assumptions. After all the calculus is driven entirely by the extensional information as expressed in the propositions (= right-hand side of :). If the calculus is to be sound then a formal consequence such as (2) allows us to infer all instances of the form (1) where $q\{p_1/x_1, \ldots, p_n/x_n\} \preccurlyeq p$. This suggests a definition of *formal entailment*

$$p_1 : \varphi_1, p_2 : \varphi_2 \ldots, p_n : \varphi_n \quad \Vdash \quad p : \psi \tag{3}$$

to mean that there exists a derivation (2) in iLC-h such that $q\{p_1/x_1, \ldots, p_n/x_n\} \preccurlyeq p : \psi$. In this section we will show that semantical (1) and formal entailment (3) coincide for the

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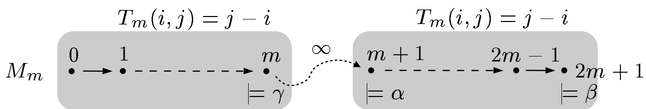
fragment of elementary propositions. The soundness direction holds for arbitrary propositions, not just elementary ones. Let us observe up front that the calculus is sound in the sense that whenever $\vdash q : \varphi$ then the λ-term q denotes an element in $[\![\varphi]\!]$. In particular, it is hereditarily monotone. After all, q can only be built from the basic monotone operations max, $+$, and 0 introduced with the axioms.

Theorem 7.1 (Intensional Soundness) *Let $\varphi_1, \ldots, \varphi_n$ and ψ be arbitrary propositions, and $p_i \in [\![\varphi_i]\!]$ and $p \in [\![\psi]\!]$ such that $\vec{p} : \vec{\varphi} \Vdash p : \psi$. Then, $\vec{p} : \vec{\varphi} \models p : \psi$.*

At this point the restriction on axiom WLC demands an explanation. Our realisation works for $[\![\zeta_1]\!] = [\![\zeta_2]\!] = \{0\}$ since then $[\![((\zeta_1 \supset \zeta_2) \supset \varphi \,\&\, (\zeta_2 \supset \zeta_1) \supset \varphi) \supset \varphi]\!] \cong [\![(\varphi \,\&\, \varphi) \supset \varphi]\!]$ is essentially the set of binary monotone functions on $[\![\varphi]\!]$. As we have seen WLC may be taken as the (polymorphic) maximum function. Now, one might wonder whether there does not in fact exist a more complex family of higher-order functions to interpret WLC that make the *unrestricted* scheme $WLC : ((\varphi_1 \supset \varphi_2) \supset \varphi \,\&\, (\varphi_2 \supset \varphi_1) \supset \varphi) \supset \varphi$ valid. Unfortunately, this is not the case. Consider the instantiation $\varphi_1 =_{df} \alpha \supset \bigcirc\beta$, $\varphi_2 =_{df} (\alpha \supset \bigcirc\beta) \supset \bigcirc\beta$, and $\varphi =_{df} \bigcirc\gamma$ for propositional atoms α, β, γ. If there existed a uniform bound for this instantiation of WLC then in particular there would exist a uniform bound p such that

$$\models \quad p : (((\alpha \supset \bigcirc\beta) \supset \bigcirc\beta) \supset \bigcirc\gamma \;\;\&\;\; (\alpha \supset \bigcirc\beta) \supset \bigcirc\gamma) \supset \bigcirc\gamma. \tag{4}$$

We show, by contradiction, that such a p cannot exist. Validity of (4) means that for every pair of functionals $F \in [\![((\alpha \supset \bigcirc\beta) \supset \bigcirc\beta) \supset \bigcirc\gamma]\!]$ and $G \in [\![(\alpha \supset \bigcirc\beta) \supset \bigcirc\gamma]\!]$ there exists a value $p\,(F, G) \in [\![\bigcirc\gamma]\!] = \mathbb{N} \times \underline{1}$ such that for all esm $M = (W, \sqsubseteq, V, T)$ and $w \in W$, with $M, w \models F : ((\alpha \supset \bigcirc\beta) \supset \bigcirc\beta) \supset \bigcirc\gamma$ and $M, w \models G : (\alpha \supset \bigcirc\beta) \supset \bigcirc\gamma$, we have $M, w \models p\,(F, G) : \bigcirc\gamma$. As a counter example take the functionals $F =_{df} \lambda f.\,(0, 0)$, $G =_{df} \lambda x.\,x\,0$, and the family of models $M_m = (\{0, 1, \ldots, 2m+1\}, \leq, V_m, T_m)$, $m \geq 0$, in which $\alpha \in V_m(i) \Leftrightarrow i \geq m+1$, $\beta \in V_m(i) \Leftrightarrow i = 2m+1$, $\gamma \in V_m(i) \Leftrightarrow i \geq m$, and $T_m(i, j) = j - i$ if $i \leq j \leq m$ or $m + 1 \leq i \leq j$, and $T_m(i, j) = \infty$ otherwise. In pictures the models look like this:



We claim that for all models, $M_m \models F : ((\alpha \supset \bigcirc\beta) \supset \bigcirc\beta) \supset \bigcirc\gamma$. For let $u \in \{0, \ldots, 2m\}$ be a world in M_m, and $f \in [\![(\alpha \supset \bigcirc\beta) \supset \bigcirc\beta]\!]$ such that $M_m, u \models f : (\alpha \supset \bigcirc\beta) \supset \bigcirc\beta$. By the structure of M_m this implies that $u \geq m+1$, regardless of f, whence $M_m, u \models (0, 0) : \bigcirc\gamma$. Thus, $M_m, u \models F\,f : \bigcirc\gamma$ as desired. Also, one can easily show that $M_m \models G : (\alpha \supset \bigcirc\beta) \supset \bigcirc\gamma$. For let $g \in [\![\alpha \supset \bigcirc\beta]\!] = \underline{1} \to \mathbb{N} \times \underline{1}$ with $M_m, u \models g : \alpha \supset \bigcirc\beta$. We distinguish two cases: If $\pi_1\,(g\,0) < m$ we must have $u > m+1$. So $M_m, u \models 0 : \gamma$, and thus $M_m, u \models G\,g : \bigcirc\gamma$ whatever $G\,g$ is. If $\pi_1\,(g\,0) \geq m$, then $\pi_1\,(G\,g) \geq m$. Since $M_m, 0 \models (m, 0) : \bigcirc\gamma$, this means in particular $M_m, u \models G\,g : \bigcirc\gamma$. So we have shown that $M_m \models F : ((\alpha \supset \bigcirc\beta) \supset \bigcirc\beta) \supset \bigcirc\gamma$ and $M_m \models G : (\alpha \supset \bigcirc\beta) \supset \bigcirc\gamma$ for all m. But on the other hand there cannot be any *fixed* $p\,(F, G) \in \mathbb{N} \times \underline{1}$ (that only depends on F, G) such that $M_m \models p\,(F, G) : \bigcirc\gamma$ for *all* $m \geq 0$. For if $p\,(F, G)$ were the pair $(n, 0)$, say, then $M_{n+1} \not\models p\,(F, G) : \bigcirc\gamma$. So, whatever the fixed value of $p\,(F, G)$ there is always a model M_m to outwit it. This shows that there is no uniform stabilisation bound for (4), and hence not for the unrestricted scheme WLC.

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Theorem 7.2 (Elementary Intensional Completeness) *Let* $\theta_1, \ldots, \theta_n$ *and* θ *be elementary propositions, and* $p_i \in [\![\theta_i]\!]$ *and* $p \in [\![\theta]\!]$ *such that* $\vec{p} : \vec{\theta} \models p : \theta$. *Then,* $\vec{p} : \vec{\theta} \Vdash p : \theta$.

We shall prove the elementary completeness Theorem in two stages. We first show completeness for situations of a special normal form and then show how every elementary situation can be reduced to normal form. To this end let us call any non-empty list $\varphi_1, \ldots, \varphi_n, \varphi$ of arbitrary propositions a **(general) problem**. If every proposition in the list $\varphi_1, \ldots, \varphi_n, \varphi$ is elementary, then we shall call the problem an **elementary problem**. We think of the φ_i as behavioural descriptions of the components of a system, and of φ as the specification of the composite system's behaviour. The semantic entailment $p_1 : \varphi_1, \ldots, p_n : \varphi_n \models p : \varphi$ then states that the composite system satisfies φ with effort bound p assuming that all components meet their specifications φ_i with effort bounds p_i. In particular, p might describe the exact bounds of the composite system. We say the calculus iLC-h is *exact* for the general problem $\varphi_1, \ldots, \varphi_n, \varphi$ if for all choices $p_i \in [\![\varphi_i]\!]$ $(i = 1, \ldots, n)$ and $p \in [\![\varphi]\!]$ such that $p_1 : \varphi_1, \ldots, p_n : \varphi_n \models p : \varphi$ we have $p_1 : \varphi_1, \ldots, p_n : \varphi_n \Vdash p : \varphi$. The Intensional Completeness Theorem 7.2 can then be restated as the claim that iLC-h is exact for all elementary problems.

7.1 Exactness for Normal Problems

We call an elementary problem *normal* if it is of the form $\chi_{\mathbb{A}}(L), \rho_1 \supset \bigcirc\sigma_1, \ldots, \rho_n \supset \bigcirc\sigma_n, \theta$ where $\chi_{\mathbb{A}}(L)$ is the characteristic proposition in atoms $\mathbb{A}$ of an irredundant *flm* L, ρ_i, σ_i families of states of L, and θ a unit ζ or a modalised unit $\bigcirc\zeta$. We also assume that that L has signature $\mathbb{A}$ and that all atoms occurring in ρ_i, σ_i, θ are contained in $\mathbb{A}$. In the following we abuse notation and denote an elementary problem $\vec{\varphi}, \psi$ by $\vec{\varphi} \Vdash \psi$.

Let $\chi_{\mathbb{A}}(L), \rho_1 \supset \bigcirc\sigma_1, \ldots, \rho_n \supset \bigcirc\sigma_n \Vdash \theta$ be a normal problem with irredundant *flm* $L = (W, \sqsubseteq, V)$ in signature $\mathbb{A}$. Then, there are worlds $r_i, s_i \in W$ such that $\rho_i = \bigwedge V(r_i)$ and $\sigma_i = \bigwedge V(s_i)$. Now suppose we are given effort bounds $p_i \in [\![\rho_i \supset \bigcirc\sigma_i]\!] = \underline{1} \to (\mathbb{N} \times \underline{1})$. Since $\underline{1} \to (\mathbb{N} \times \underline{1}) \cong \mathbb{N}$ we are free to consider the bounds p_i as natural numbers, although strictly speaking, they are functions. We will apply the same confusion more generally to bounds $d \in [\![\bigwedge V(x) \supset \bigcirc \bigwedge V(y)]\!] \cong \mathbb{N}$, for arbitrary $x, y \in W$. We may view the intensional theory

$$\vec{p} : \Theta \quad =_{df} \quad 0 : \chi_{\mathbb{A}}(L), p_1 : \rho_1 \supset \bigcirc\sigma_1, \ldots, p_n : \rho_n \supset \bigcirc\sigma_n$$

as the canonical specification of a particular intensional enrichment of L in the following way. First we observe that every transition $\rho_i \supset \bigcirc\sigma_i$ amounts to a boundedness constraint for the effort between state r_i and s_i. If we want to know the tightest upper bound for the transition from some state $w \in W$ to some other state $v \in W$, $w \sqsubseteq v$, then we need to find the minimal element in the set $C(w, v) =_{df} \{ d \mid \vec{p} : \Theta \vdash d : \bigwedge V(w) \supset \bigcirc \bigwedge V(v) \}$. Each $d \in C(w, v)$ is a provable upper bound on the effort expended in the interval $[w, v)$ in theory $\vec{p} : \Theta$. Let us call the minimal element $\delta(w, v) =_{df} \mathit{min}\ C(w, v)$ in this set the **formal effort** of v from w in theory $\vec{p} : \Theta$. Note that if $C(w, v) = \emptyset$ then the step from w to v is unconstrained by $\vec{p} : \Theta$, in which case we may put $\delta(w, v) = \infty$. The definition implies that if $\delta(w, v) < \infty$ then there must exist a proof $\vec{p} : \Theta \vdash \delta(w, v) : \bigwedge V(w) \supset \bigcirc \bigwedge V(v)$. Now, let $L^\delta =_{df} (W, \sqsubseteq, V, \delta)$ be the linear model L given formal effort measure δ. One can show that L^δ is the canonical model

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