A Metzlerian business cycle model with nonlinear heterogeneous expectations

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Abstract

Metzler's model is an important contribution to our understanding of the dynamics of business cycles. In this model, the production of consumption goods depends on expected future sales. However, Metzler assumes that producers are homogeneous and follow a simple expectation formation rule. Taking into account that in reality producers might not only follow several expectation formation rules, but might also even switch between them, we reformulate Metzler's original model. Endogenous business cycles may emerge within our model, i.e. changes in production and inventory are (quasi-)periodic for certain parameter combinations.

Keywords

business cycles, inventory adjustment, heterogeneous expectations, nonlinear dynamics, Neimark-Sacker bifurcation

1 Introduction

The goal of our paper is to show that the expectation formation of boundedly rational heterogeneous firms may cause endogenous business cycle dynamics. As a workhorse, we rely on Metzler's (1941) inventory model in which producers use a single behavioral expectation formation rule to predict future sales of consumption goods. In case producers mispredict actual sales, their inventories deviate from desired levels. Metzler's contribution is to show

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that the inventory adjustments of firms may lead to dampened business cycles (see Gandolfo (2005) for a comprehensive analysis of this model). Due to its strength, Metzler's inventory approach is quite popular and has been extended in various directions; see, e.g. Eckalbar (1985), Franke and Lux (1993), Matsumoto (1998) or Chiarella et al. (2005).

We believe that Metzler's model may also be regarded as a very natural environment to study the interplay between firms' expectation formation behavior and the evolution of national income.[1] In reality, firms obviously have to forecast their future sales in order to determine their production level. Experimental evidence indicates that human agents are boundedly rational and tend to rely on simple heuristics when predicting the future (see, e.g. Simon (1955), Tversky and Kahneman (1974), Smith et al. (1988)). Within our model, producers thus select between different types of predictors. To make matters as simple as possible, they have the choice between an extrapolative and a regressive predictor to forecast future sales. Furthermore, we assume that producers select between these rules with respect to market circumstances. For instance, if a boom or slump is relatively pronounced, more and more agents may come to the conclusion that a reversion to equilibrium output is overdue and may therefore switch to the regressive expectation formation rule.

Using a mixture of analytical and numerical methods to analyze our nonlinear dynamical model, we are able to derive the following results.[2] Our model has a unique fixed point that corresponds to the Keynesian multiplier solution. This fixed point may only lose its local asymptotic stability via a (so-called supercritical) Neimark-Sacker bifurcation, after which (quasi-)periodic motion sets in. The parameter space that ensures local asymptotic stability of the fixed point negatively depends on the strength with which firms extrapolate future sales. In contrast, the parameter of the regressive predictor has a stabilizing impact on the amplitude of business cycles. Overall, we thus conclude that the expectation formation of boundedly rational heterogeneous producers may in fact be an important source of endogenous upswings and downturns in economic activity.

The paper is organized as follows. Our model is presented in section 2, followed by a derivation of analytical results in section 3. Section 4 contains a numerical illustration of how functions our model. The final section provides

[1] Our paper is related to Westerhoff (2006), Lines and Westerhoff (2006) and Lines (2007) who introduce heterogeneous expectation formation into Samuelson's (1939) multiplier-accelerator model. Their approaches are able to produce complex business cycle dynamics.

[2] For recent nonlinear goods market-based business cycle models see, e.g. Puu et al. (2005) and Puu (2007).

some conclusions.

2 A reformulation of Metzler's model

In Metzler's business cycle model, national income is determined by the total production of firms which, in turn, is equal to the sum of consumption goods produced for sale U, the production of consumption goods for stock S and the production of investment goods I. At any time t national income Y can therefore be written as

$$Y_t = I_t + S_t + U_t. \tag{1}$$

To simplify matters, production of investment goods is assumed to be fixed, i.e.

$$I_t = \overline{I}. \tag{2}$$

Producers need to adjust their inventory stock at every period. If $\widehat{Q}_t$ denotes the desired level of the inventory at time t and Q_{t-1} denotes the realized inventory at the close of the previous period, production for inventory purposes is simply given by

$$S_t = \widehat{Q}_t - Q_{t-1}. \tag{3}$$

The desired level of the inventory is considered to be positively related to the expected sale of consumption goods, so that with a certain factor $k > 0$ we have

$$\widehat{Q}_t = kU_t. \tag{4}$$

In turn, the realized inventory at time t is determined by the desired level of the inventory of the previous period adjusted by unexpected inventory changes. Since unexpected inventory changes are simply the difference between realized and expected sales, we obtain

$$Q_{t-1} = \widehat{Q}_{t-1} - (C_{t-1} - U_{t-1}). \tag{5}$$

Realized sales are equal to consumption C, which is proportional to the national income Y. Let $0 < b < 1$ be the marginal propensity to consume. Then consumption at time t is given as

$$C_t = bY_t. \tag{6}$$

Production of consumption goods for sale depends on expected future sales. Metzler (1941), for instance, does in fact assume that producers only apply a single behavioral expectation formation rule. For instance, all producers may form extrapolative expectations.

The goal of our paper is to broaden this view. In contrast to Metzler, we assume that producers form heterogeneous expectations about future sales, they can even change their mind and switch from one expectation rule to another. To simplify matters, we only focus on two different expectation formation rules, denoted by U_t^E and U_t^R. Producers of group one are characterized by extrapolating the trend in consumption. If consumption C in period $(t-1)$ is above (below) the equilibrium level of consumption $\overline{C}$, producers in this group believe consumption will rise (fall). The following equation describes this extrapolative expectation rule

$$U_t^E = C_{t-1} + c(C_{t-1} - \overline{C}), \tag{7}$$

where $c \geq 0$ denotes the expected speed of deviation from equilibrium $\overline{C}$. Note that for $c = 0$, (7) incorporates naive expectations.

In contrast, producers in the second group follow a regressive expectation rule. They believe that consumption will always return to its equilibrium value in the long run. Formally, the prediction rule of this group is given by

$$U_t^R = C_{t-1} + f(\overline{C} - C_{t-1}), \tag{8}$$

where $0 \leq f \leq 1$ denotes the adjustment speed towards equilibrium. With $f = 1$, adjustment is expected to take place instantly, while for $f = 0$ producers in this group form naive expectations.

The expected sales of consumption goods U_t can now be expressed as a weighted average of the expectations of these two groups:

$$U_t = w_t U_t^E + (1 - w_t) U_t^R. \tag{9}$$

The weight given to trend-extrapolating producers in a certain period t is described as

$$w_t = \frac{1}{1 + d(\overline{C} - C_{t-1})^2}. \tag{10}$$

The intuition of this switching mechanism is quiet clear: If consumption in period t is close to equilibrium consumption, most producers will extrapolate the trend in consumption. However, the further consumption drifts away from equilibrium, the more producers are convinced that the current boom or slump will end, and will subsequently switch to regressive expectations.[3] The weighting function is therefore bell-shaped. Parameter d captures the popularity of regressive expectations. The higher d is, the more agents rely on regressive expectations (for a given distance between C and $\overline{C}$).

Combining (1) - (10) reveals that

$$Y_t = h(Y_{t-1}, Y_{t-2}), \tag{11}$$

[3] Obviously, business cycles imply that booms and slumps will eventually burst.

i.e. national income is determined by a second-order nonlinear difference equation.[4]

3 Some analytical results

In order to calculate the fixed points of our model, we consider the following simple transformation. Combining (1) - (5) leads to

$$Y_t = U_t + kU_t - (1+k)U_{t-1} + C_{t-1} + \overline{I}. \tag{12}$$

From this equation it is easy to see that our model has only one fixed point $\overline{Y}$, which is given by

$$\overline{Y} = \frac{1}{1-b}\overline{I}. \tag{13}$$

Recall that (13) corresponds to the well-known Keynesian multiplier solution. In the appendix we show that the necessary and sufficient conditions for local asymptotic stability are

$$1 + b(3 + 2k + 2c(1+k)) > 0 \tag{14}$$

$$1 - b > 0 \tag{15}$$

$$1 - b(1+c)(1+k) > 0. \tag{16}$$

Due to parameter restrictions, conditions (14) and (15) are always satisfied so that only the last of the above three conditions may be violated.
Closer scrutiny of (16) reveals that the fixed point is locally asymptotically stable if

$$k < \frac{1-b-bc}{b(1+c)}. \tag{17}$$

Figure 1 illustrates the relation between b and k for different values of c. The fixed point is locally asymptotically stable for parameter combinations of b and k below the graph, while the fixed point is unstable for combinations above the graph. Note that the higher parameter c is, the easier one can find realistic combinations for b and k for which the fixed point is no longer locally asymptotically stable. If c is close to unity, for example, a relatively small marginal propensity to consume amounting to $b = 0.5$ can even destroy local asymptotic stability if k is very small and close to zero.

[4] We do not track the individual investment and inventory relations of firms. However, in the appendix we demonstrate that from an aggregated point of view, relations (2) and (3) still hold.

— Fig. 1 goes about here —

Also note that parameters d and f are irrelevant to the local asymptotic stability of the fixed point, i.e. stability is completely independent of both the popularity of regressive expectations and the (estimated) adjustment speed of the regressive expectation formation rule.
Finally, violation of (16) is a necessary condition for the emergence of a Neimark-Sacker bifurcation (see, e.g. Medio and Lines (2001)). As supported by numerical evidence in the next section, the bifurcation is of the supercritical type, i.e. if (16) is violated, the fixed point not only loses its local asymptotic stability but also a stable invariant curve appears and production becomes (quasi-)periodic.

4 Some numerical results

Let us first illustrate the dynamics our model may produce. Figure 2 shows a simulation run for $b = 0.75$, $c = 0.3$, $d = 1$, $f = 0.1$, $k = 0.1$ and $\overline{I} = 10$. With this parameter setting the equilibrium values of income and consumption are $\overline{Y} = 40$ and $\overline{C} = 30$.

— Fig. 2 goes about here —

Dynamics can now be explained as follows: in time period $t = 1$ income Y is above its equilibrium value. As long as this deviation from equilibrium is small, the majority of producers do indeed believe that income and sales will rise. This implies not only a relatively large weight of extrapolative expectations, but also induces the production of consumption goods for sale to increase. Since the desired inventory level is proportional to expected sales, producers also wish to maintain a higher inventory level. From now on, dynamics become increasingly self-fulfilling. As producers extend the production of consumption goods for sale and inventory purposes, income Y will rise. An increase in income Y immediately leads to an increase in consumption C which, in turn, triggers producers' expectations of a further rise of income and sales.
That said, the more income exceeds its equilibrium value, the more producers become pessimistic. They make up their minds and start believing that income and consumption will return to their equilibria. As more and more producers now follow the regressive expectation rule, growth of expected sales of consumption goods slows down. If $(1-w)$ reaches a certain critical value of about 0.2, expected sales themselves will decrease. Producers now wish to realize a lower inventory level and start reducing the stock. They disinvest, and

income starts to fall. As the distance between C and $\overline{C}$ decreases, however, more and more producers will again hold extrapolative expectations about future sales. However, due to the depletion of the inventory stock, income and related consumption now have fallen below their equilibria. Most producers will therefore extrapolate a negative trend in consumption and expect consumption to decrease further. As a result, income and consumption will fall increasingly below their equilibria until the recession becomes too large, and most producers again expect consumption to return to its equilibrium.
The periodic motion continues, and business cycles with upswings and downturns are created endogenously within our model. Note that the production of consumption goods U and inventory Q generally pass identical cycles to income and consumption, but that these movements are time-lagged.
The fixed point of our model may only lose its local asymptotic stability via a Neimark-Sacker bifurcation. With our initial parameter setting given above, the critical values for a Neimark-Sacker bifurcation to occur (when all other parameters fixed) are given by

$$b^* = 0.699, \tag{18}$$

$$c^* = 0.212, \tag{19}$$

$$k^* = 0.026, \tag{20}$$

respectively.
Figure 3 displays the bifurcation diagrams of the corresponding parameters. Bifurcation diagrams are a powerful tool to analyze the dynamics of nonlinear systems. Each parameter is increased in 200 discrete steps, as indicated on the axis, while all other parameters remain fixed. 100 observations are plotted for each parameter value. A longer transient phase of about 1000 steps is omitted.

— Fig. 3 goes about here —

The results of this exercise are as follows: first, our analytical predictions are accurately confirmed. A (supercritical) Neimark-Sacker bifurcation becomes clearly visible at the calculated values. Our results thereby suggest that (quasi-)periodic behavior does not set in immediately or abruptly, but that the path towards periodic and cyclical behavior is rather smooth. The latter result eminently applies to parameter b. Furthermore, the amplitude of business cycles seems to increase with b, c and k. This result especially applies to parameter c. Trend-extrapolating producers may therefore be regarded as a source of instability.

As we have seen so far, stability of the fixed point is independent of parameters d and f. However, this does not mean that these parameters do not influence dynamics at all. Figure 4 shows the corresponding bifurcation diagrams for d and f. Each parameter is increased from zero to unity in 200 discrete steps.

— Fig. 4 goes about here —

The bifurcation diagrams do not only confirm that the stability of the fixed point is independent of parameters d and f but also reveal that the amplitude of the business cycle declines as the parameters increase. The more producers favor the regressive expectation formation rule and the higher producers with such expectations estimate the adjustment speed towards equilibrium, the lower the amplitude of the business cycle. It follows from this that producers who follow the regressive expectation formation rule tend to stabilize dynamics.

5 Conclusion

In this paper we reformulate Metzler's well-known business cycle model by allowing producers to select between different types of expectation formation rules. In reality, producers are not at all homogeneous and may hold different expectations about future sales of consumption goods. With this modification, we obtain the following results:
Our model has a unique fixed point. This fixed point may only lose its local asymptotic stability via a (supercritical) Neimark-Sacker bifurcation, after which (quasi-)periodic motion sets in. Business cycles are thereby endogenously created within the model. No exogenous shocks are necessary to explain the upswings and downturns of national income and related variables such as production and inventory.
Moreover, our results suggest that extrapolating producers are a source of instability, while producers with regressive expectations tend to stabilize dynamics. The higher extrapolating producers regard the speed of deviation from equilibrium, the easier one can find realistic combinations of marginal propensity to consume and the desired inventory level for which the fixed point is no longer stable.

A Stability of the fixed point

By introducing the auxiliary variable $Z_t = Y_{t-1}$, (11) may be rewritten as a first-order nonlinear difference equation system

$$\begin{aligned} Y_t &= h(Y_{t-1}, Z_{t-1}) \\ Z_t &= g(Y_{t-1}, Z_{t-1}) \end{aligned} \tag{21}$$

The Jacobian matrix J of the *linearized* difference equation system is determined by the derivatives of (21), i.e.

$$J = \begin{pmatrix} \frac{\partial h}{\partial Y_{t-1}} & \frac{\partial h}{\partial Z_{t-1}} \\ \frac{\partial g}{\partial Y_{t-1}} & \frac{\partial g}{\partial Z_{t-1}} \end{pmatrix} \tag{22}$$

Note that $\frac{\partial g}{\partial Y_{t-1}} = 1$ and $\frac{\partial g}{\partial Z_{t-1}} = 0$.
Inserting the fixed point $\overline{Y} = Y_t = Y_{t-1} = \frac{\overline{I}}{1-b} = Z_t = Z_{t-1} = \overline{Z}$ into (22) leads to

$$\widetilde{J} = \begin{pmatrix} b(2+c+k+ck) & -b(1+c)(1+k) \\ 1 & 0 \end{pmatrix} \tag{23}$$

$\widetilde{J}$ denotes the Jacobian matrix calculated at the fixed point. The trace and determinant of this matrix are simply

$$tr(\widetilde{J}) = b(2+c+k+ck) \tag{24}$$

$$det(\widetilde{J}) = b(1+c)(1+k). \tag{25}$$

According to Medio and Lines (2001), the necessary and sufficient conditions guaranteeing that the fixed point of our second-order nonlinear difference equation is locally asymptotically stable are

$$1 + tr(\widetilde{J}) + det(\widetilde{J}) > 0 \tag{26}$$

$$1 - tr(\widetilde{J}) + det(\widetilde{J}) > 0 \tag{27}$$

$$1 - det(\widetilde{J}) > 0. \tag{28}$$

Finally, combining (24) - (28) leads to

$$\begin{aligned} 1 + b(3 + 2k + 2c(1+k)) &> 0 \\ 1 - b &> 0 \\ 1 - b(1+c)(1+k) &> 0. \end{aligned}$$

B Aggregated investment and inventory relations

The aggregated production of investment goods at time t may be written as the following weighted average

$$I_t = w_t I_t^E + (1 - w_t) I_t^R, \tag{29}$$

where I_t^E and I_t^R denote the production of investment goods for producers who either follow the extrapolative or the regressive expectation formation rule.
As the production of investment goods is considered fixed for all producers, (29) simplifies to

$$I_t = w_t \overline{I} + (1 - w_t) \overline{I} = \overline{I},$$

i.e. aggregated production of investment goods is equal to (2).
Due to market circumstances, producers may change their mind and switch from one expectation formation rule in period $(t-1)$ to another in period t. Let us suppose that more producers switch from the regressive to the extrapolative expectation formation rule. The aggregated production S of inventory goods in period t may then be expressed as

$$\begin{aligned} S_t = {} & w_{t-1} \left(k U_t^E - (k-1) U_{t-1}^E + C_{t-1} \right) + (1 - w_t) \left(k U_t^R - (k-1) U_{t-1}^R + C_{t-1} \right) \\ & + (w_t - w_{t-1}) \left(k U_t^E - (k-1) U_{t-1}^R + C_{t-1} \right). \end{aligned}$$

Straightforward calculations now reveal that

$$\begin{aligned} S_t &= k \left(w_t U_t^E + (1 - w_t) U_t^R \right) - (k-1) \left(w_{t-1} U_{t-1}^E + (1 - w_{t-1}) U_{t-1}^R \right) + C_{t-1} \\ &= k U_t - (k-1) U_{t-1} + C_t \\ &= \widehat{Q}_t - Q_{t-1}, \end{aligned}$$

i.e. producers who switch from one expectation formation rule to another do not change inventory production from an aggregated point of view.

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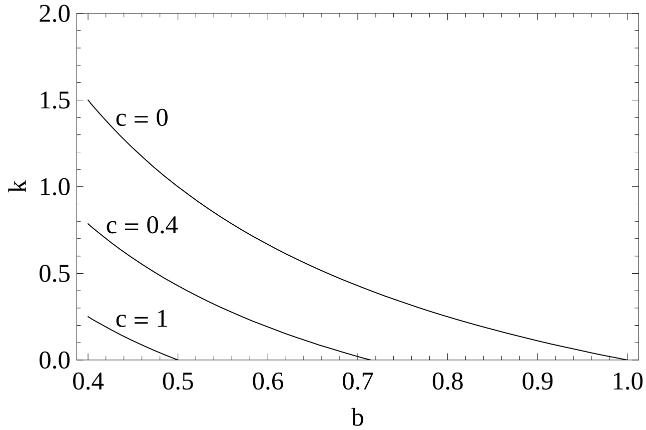

Figure 1: Local asymptotic stability areas in (k, b) parameter space for different values of c. Combinations below (above) the graphs imply a locally asymptotically stable (unstable) fixed point.

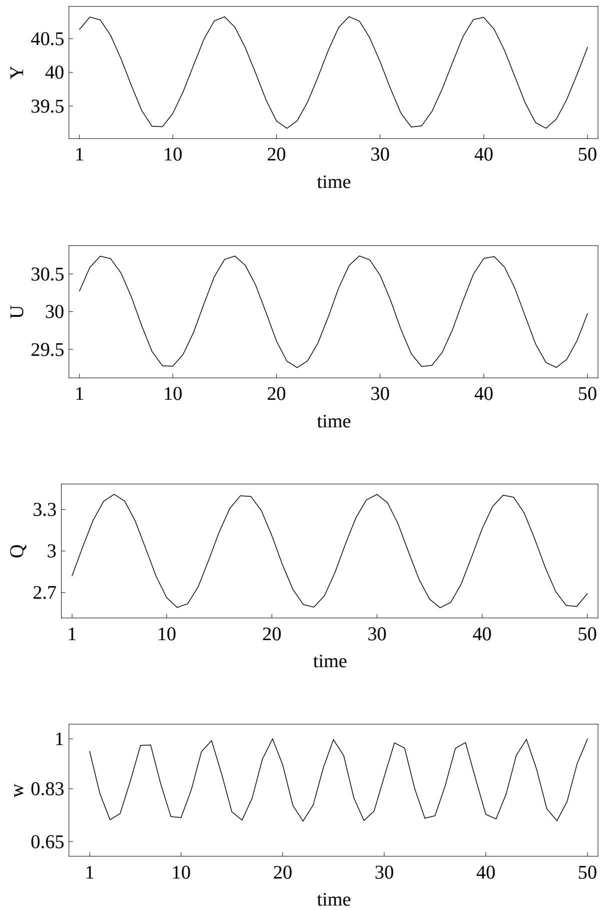

Figure 2: Evolution of national income Y, expected sales U, inventory Q and weight w of extrapolative expectations in a simulation run for $b = 0.75$, $c = 0.3$, $d = 1$, $f = 0.1$, $k = 0.1$ and $\overline{I} = 10$. Equilibrium values of income and consumption are $\overline{Y} = 40$ and $\overline{C} = 30$.

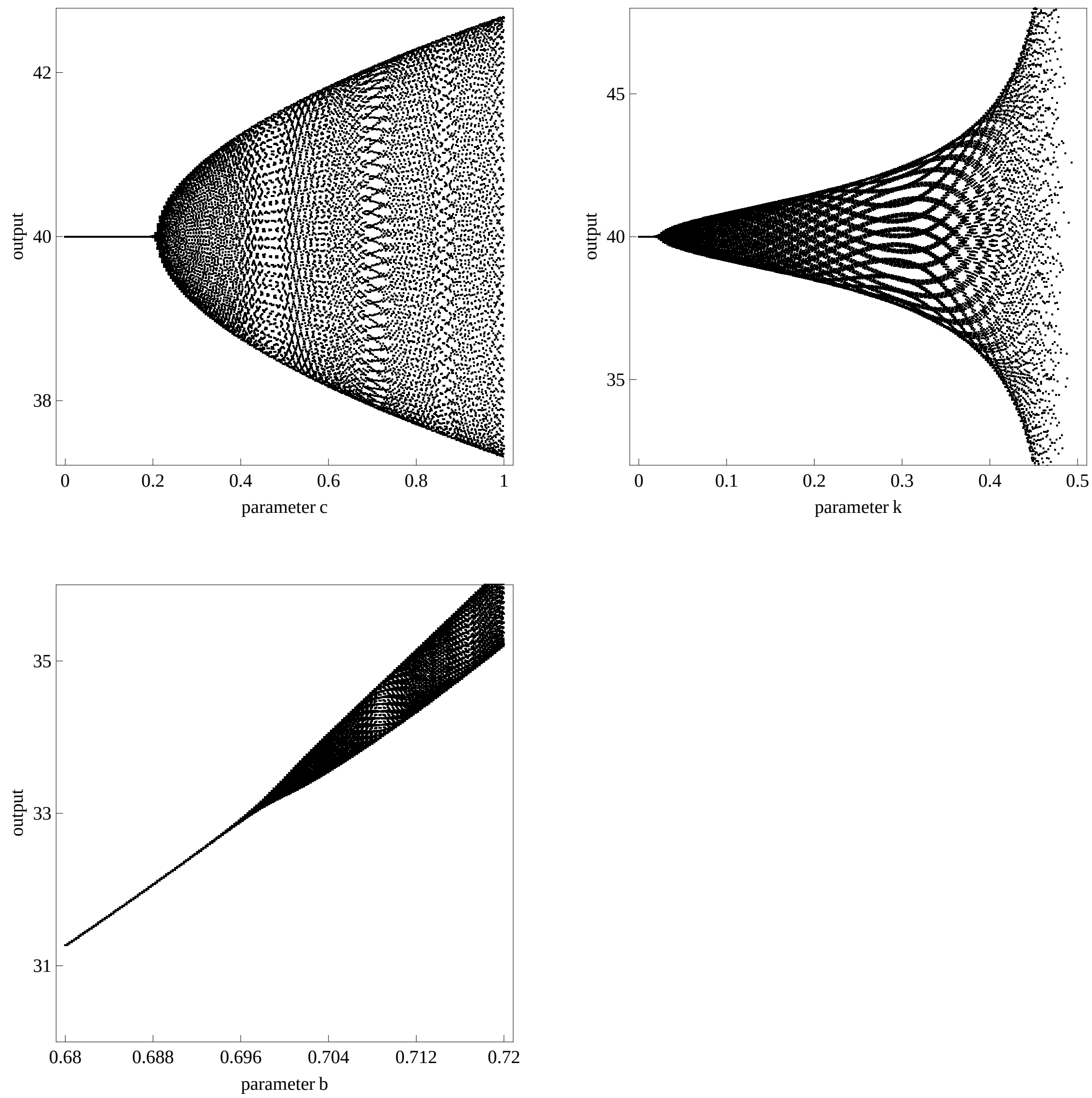

Figure 3: Bifurcation diagrams for parameters c, k and b. Each parameter is increased in 200 discrete steps. Dynamics are plotted after a transient of 1000 observations. The other parameters are identical to those in figure 2.

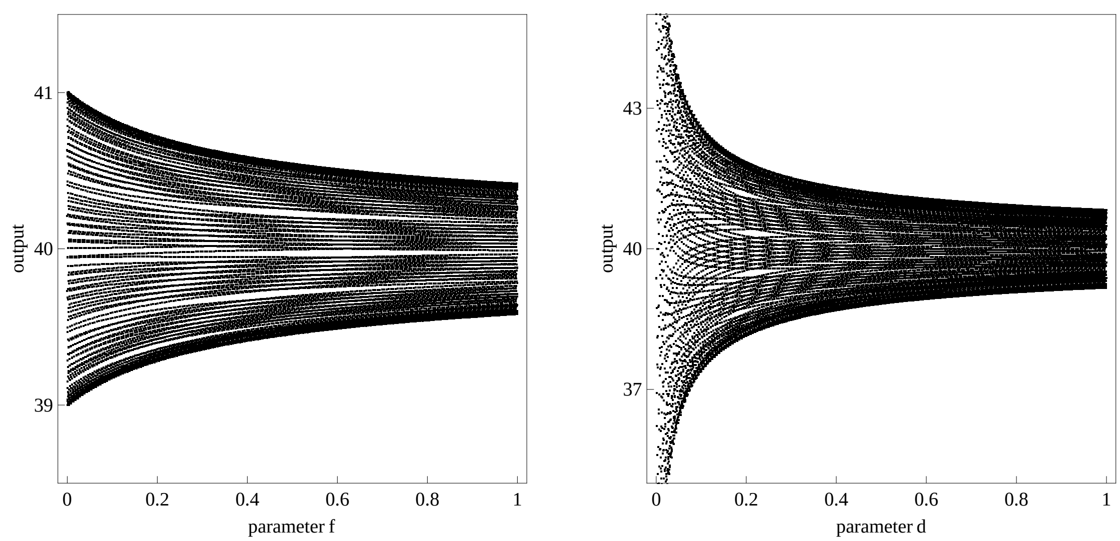

Figure 4: Bifurcation diagrams for parameters d and f. Each parameter is increased from zero to unity in 200 discrete steps. Dynamics are plotted after a transient of 1000 observations. The other parameters are identical to those in figure 2.