Paradox of simple limiter control

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Chaos control by simple limiters is an easy-to-implement and effective method of stabilizing irregular fluctuations. Here we show that applying limiter control to a state variable can significantly shift its mean value. In many situations, this is a countereffective as well as unexpected result, when the aim of control is also to restrict the dynamics. We discuss this effect on the basis of a model of population dynamics and conclude that it can have severe implications for the management of pest species and epidemic spread.

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The control of chaotic systems is an ongoing topic of research in physics as well as in related fields of practical applications—e.g., in mechanical systems, electronic systems, chemical systems, neural networks, or heart tissue [1,2]. One method of chaos control is simple limiters, which restrict the evolution of a state variable in a certain direction [3,4]. Due to their simplicity, they need neither detailed knowledge of the system's state nor time-consuming generation of a control signal, which makes them especially suitable for high-speed systems [3–6]. They have also been suggested for regulating cardiac rhythms [7], adaptive dynamics [8], arithmetic computations [9], commodity markets [10,11], neuronal systems, and smart matter applications (reviewed in Ref. [12]). Theoretical investigations can be found in Refs. [13–16]. Here we report that the application of limiters may have counterintuitive effects. We illustrate this paradoxic effect by means of the quadratic map as a simple model of population dynamics and show that well-intended management measures for the control of stock farming, pest species, or epidemics may be countereffective. For instance, implementing an upper level for the population dynamics may result in a boost of population size (rather than in a reduction as one may expect at first sight).

Chaos is composed of an infinite number of unstable periodic orbits. By applying a limiter on a single, fixed level, the dynamical range of a chaotic oscillator can be bounded. Thus, one of the unstable periodic orbits can be stabilized with only small control perturbations. The properties of limiter control are fully described by one-parameter one-dimensional flat-topped maps [13]. For a one-dimensional discrete map $f(x_n)$ with x_n denoting the state variable at time n, the control scheme with a limiter value h is implemented by

$$x_{n+1} = \min\{f(x_n), h\}. \qquad (1)$$

In a series of classical papers in the mid-1970s, May [17–19] showed that the quadratic map

$$f(x_n) = r(1 - x_n)x_n, \qquad (2)$$

with $x_0 \in (0,1)$, as well as other difference equations arising in population biology, has a rich dynamical structure including chaos. He thus introduced the ideas of nonlinear dynamics to a broader spectrum of science and to ecology in particular (for recent monographs see Refs. [20,21]). Equation (2) can be regarded as a model of density-dependent growth for a population with nonoverlapping generations. The parameter $r \in (0,4]$ describes the intrinsic growth rate, and the carrying capacity has been scaled to unity. Increasing r as a control parameter, the quadratic map undergoes the well-known cascade of period-doubling bifurcations leading to chaos.

The concept of chaos control is also of relevance in biology (some examples are reviewed in Ref. [22]). Actually, constant feedback control, which was introduced by Parthasarathy and Sinha [23] into the physics literature, had already been applied in an ecological context by McCallum [24], who found that the chaotic population dynamics changes to simple cyclical behavior in a wide parameter range if a simple constant of external recruitment is added. This recognition was picked up by Stone [25], who showed that a small perturbation is enough to break down the period-doubling route to chaos. Applications of constant feedback control in ecological models (usually interpreted as migration, reservoir, harvest, or depletion) can also be found in Refs. [26–32]. Doebeli [33] applied certain adjustments to the growth rate, thus driving the population to a stable state. The proportional feedback method [34] has been investigated in the context of metapopulations [35] as well as in continuous-time and individual-based models of population dynamics [36]. Various control schemes have been explored by Gamarra *et al.* [37]. Desharnais *et al.* [38] developed another control scheme based on "hot-spot" regions in the state space (measured by the Lyapunov exponent) and experimentally applied it to a laboratory population of the flour beetle *Tribolium castaneum*.

In biological terms, the application of a simple limiter to the quadratic map corresponds to control measures such as culling of a stock population, hunting or catching of a managed population and stock, eradicating pest species, or treating infectious diseases. Figure 1 shows the bifurcation diagram of the quadratic map in the chaotic range, when a limiter is applied. For $h=1$, the dynamics is uncontrolled and chaotic, but already for any slightly smaller $h<1$ the oscillations become periodic [7,15,39]. At first, there are many cycles, but then the number of cycles reduces. At around

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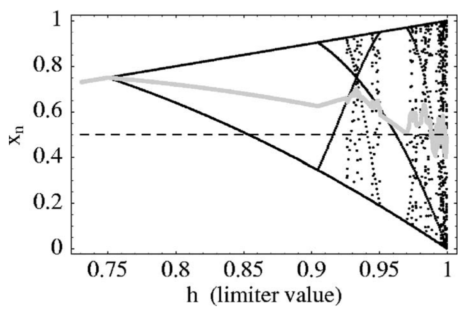

FIG. 1. Paradox of simple limiter control. Limiters force the chaotic dynamics of the quadratic map (1), (2) with $r=4$ to periodic cycles or stable fixed points. The mean of the asymptotic state variables (gray line) is for a wide range of limiter values larger than the mean of the quadratic map without limiter control (dashed line).

$h=0.9$ there appears a two-cycle, which amplitudes decrease with h and vanish at $h=0.75$. There, the limiter value coincides with the nontrivial fixed point. For smaller limiter values $h<0.75$, the dynamics is simply forced to the limiter.

The dashed line in Fig. 1 corresponds to the mean value of the state variable in the uncontrolled chaotic regime, while the gray line gives the mean values $\bar{x}_n$ of the controlled dynamics. For high values of h close to 1, deviations of $\bar{x}_n$ from the uncontrolled mean value are relatively small. This is probably due to the fact that the oscillations still have many cycles which range almost over the entire unit interval. At the other end of the limiter values—i.e., $h<0.75$—$\bar{x}_n$ trivially decreases linearly with the limiter value. The most interesting part is the range in between, in which $\bar{x}_n$ significantly increases—up to a maximum of 150% at $h=0.75$. The extent of this boost depends on the number and location of the cycle points and their branches. Obviously, the number of cycles in this range is smaller than for high limiter values. Furthermore, it should be noted that the smallest point of the cycles increases faster with decreasing h than the largest one gets smaller. This particularly explains why $\bar{x}_n$ increases in the range of the two-cycle.

Limiter control is effective in stabilizing chaotic orbits, but the jumping mean value of the controlled state variable may be in many applications an unexpected and undesirable effect. Imagine, for example, you are a stock farmer and wish to limit your population—e.g., in order to avoid losses due to overcrowding, increased intraspecific competition, or disease spread. This is effective only for a very small range of control perturbations with limiter values being nearly unity. In contrast, for the vast majority of limiters, the mean population size boosts and exhibits an unintended outbreak. The aim of limiting the population growth is apparently countereffective. Hence, management decisions can thus have strongly opposite impacts. We call this effect the *paradox of limiter control*.

An intuitive explanation of this paradox is given in Fig. 2. The well-known cobwebbing algorithm is applied to both the quadratic map and its limiter-controlled variant. The algorithm starts at $x_0=0.5$, because from there the maximum of the logistic map is reached. This allows the dynamics to be mapped back to the descending branch intersecting the abscissa and illustrates that the whole interval $[0,1]$ can be reached. (It should be noted, however, that for nonzero solutions r has to be slightly changed or that small perturbations around $x_n=0.5$ are necessary.) Conversely, in the limiter-controlled model the mapping cannot explore the whole descending branch, because the top is cut off. The dynamics is thus restricted to a much smaller interval. The differences in possible densities are highlighted in thick gray. One can easily see that the missing lower interval is a larger one than the missing upper interval. This means that the state interval is "shifted up," which explains the larger mean density. We

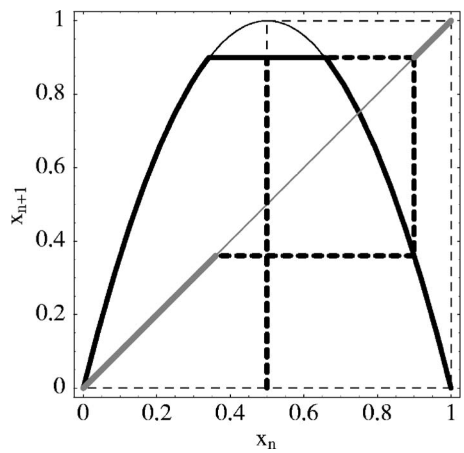

FIG. 2. The limiter control cuts off the top of the quadratic map (thin line), thus forcing the dynamics to remain within a restricted state interval (thin diagonal). The nonaccessible lower interval is larger in extent than the nonaccessible upper interval (both in gray lines), which helps to explain the paradox of limiter control by a "shift-up" of the possible state space. The cobwebbing is displayed in respective dashed lines. Parameters: $r=4$, $h=0.9$, and $x_0=0.5$.

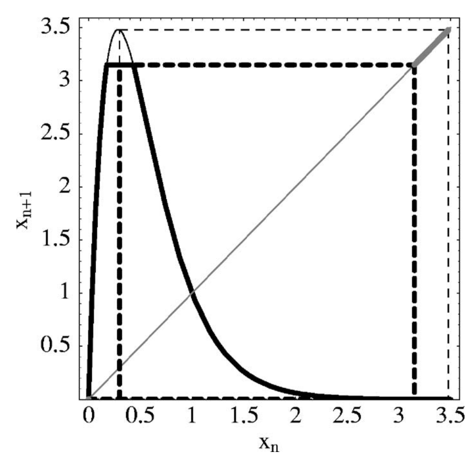

FIG. 3. Unimodel maps with a descending branch that becomes convex after a turning point have a wide range where they asymptotically approach zero. Cutting off the top does not prevent the dynamics from reaching small values. There is almost no restriction of possible state space at the lower interval (contrary to the quadratic map in Fig. 2), because of which the paradoxic effect of limiter control cannot be observed. This example is the Ricker map $f(x_n)=x_n\exp[r(1-x_n)]$ with parameters $r=3.5$, $h=3.15$, and $x_0=0.3$.

expect the paradox of limiter control to occur in flat-topped unimodel maps, the descending branch of which does not turn around (as, for example, in the tent map). In turn, maps with a turning point and a convex segment asymptotically approaching zero for large state variables will presumably not show this paradoxic effect. As an example, this is illustrated for the Ricker map [40] in Fig. 3. Since the convex branch exhibits small mapping values for all larger x_n, the image x_{n+1} can be mapped back again also to the center of the mapping's top. Consequently, there is almost no restriction for small values of the state variable. As highlighted by the gray line, the controlled dynamics cannot access an upper part of the state interval, but simulations indicate that the means of the controlled and uncontrolled versions do not differ much from each other.

The paradox of limiter control as exemplified with the quadratic map has important consequences for the management of populations. As a conclusion, we warn against the naive application of unreflected measurement programs. Well-intended perturbations (limiting a population stock) can be highly countereffective (population outbreak). Similar paradoxic effects are known in the ecological literature. E.g., an increase of the carrying capacity destabilizes the population towards densities prone to extinction (paradox of enrichment) [41], a predator mediates the coexistence of competing prey species which otherwise would go extinct [42], and the eradication of invaders that were threatening endemic species causes a much greater harm to the latter (mesopredator release) [43,44].

The caveat of limiter control may be of relevance in the control of pest species, biological invasions, and the spread of infectious diseases as well. For example, Eq. (2) arises in standard discrete-time epidemiological models of SI-(susceptible→infected) and SIS-type (as SI but with recovery of infected to the susceptible state again), where x_n describes the infected part of a population [45]. Limiting the disease spread—e.g., by medical treatment or quarantine—could thus increase the prevalence of infection. There are also many other fields of applications for the quadratic map—e.g., in genetics, economics, or social sciences (see Ref. [19] and references therein). Hence, the lesson from this study is once more that decision makers should carefully assess possible control methods. Moreover, modeling can be a helpful and powerful tool in this process.

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