# THE DOŠEN SQUARE UNDER CONSTRUCTION: A TALE OF FOUR MODALITIES*

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# What is this about?

- Constructive interpretation of the modal square of oppositions involving `positive' + `negative' modalities: necessary ($\Box$), possible ($\Diamond$), unnecessary ($\boxminus$), impossible ($\diamondminus$)
- Constructive Modal Logic CKD ("*Constructive K á la Došen*")
  - conservative extension of intuitionistic propositional logic IPL
  - first constructive logic combining all 4 modalities
- Model Theory + Proof Theory of CKD
  - Bi-relational Kripke frames
  - Hilbert calculus (HCKD)
  - Gentzen-Dragalin multi-conclusion sequent calculus (GCKD)

# PLAN OF THE TALK

1. Introduction
2. Syntax and Intuitionistic Semantics
3. Constructive Modalities CKD
4. Hilbert Deduction (HCKD) (… CKD Theories)
5. Gentzen Sequent Calculus (GCKD) (… The Došen Square at Work)
6. Conclusions

# 1 INTRODUCTION

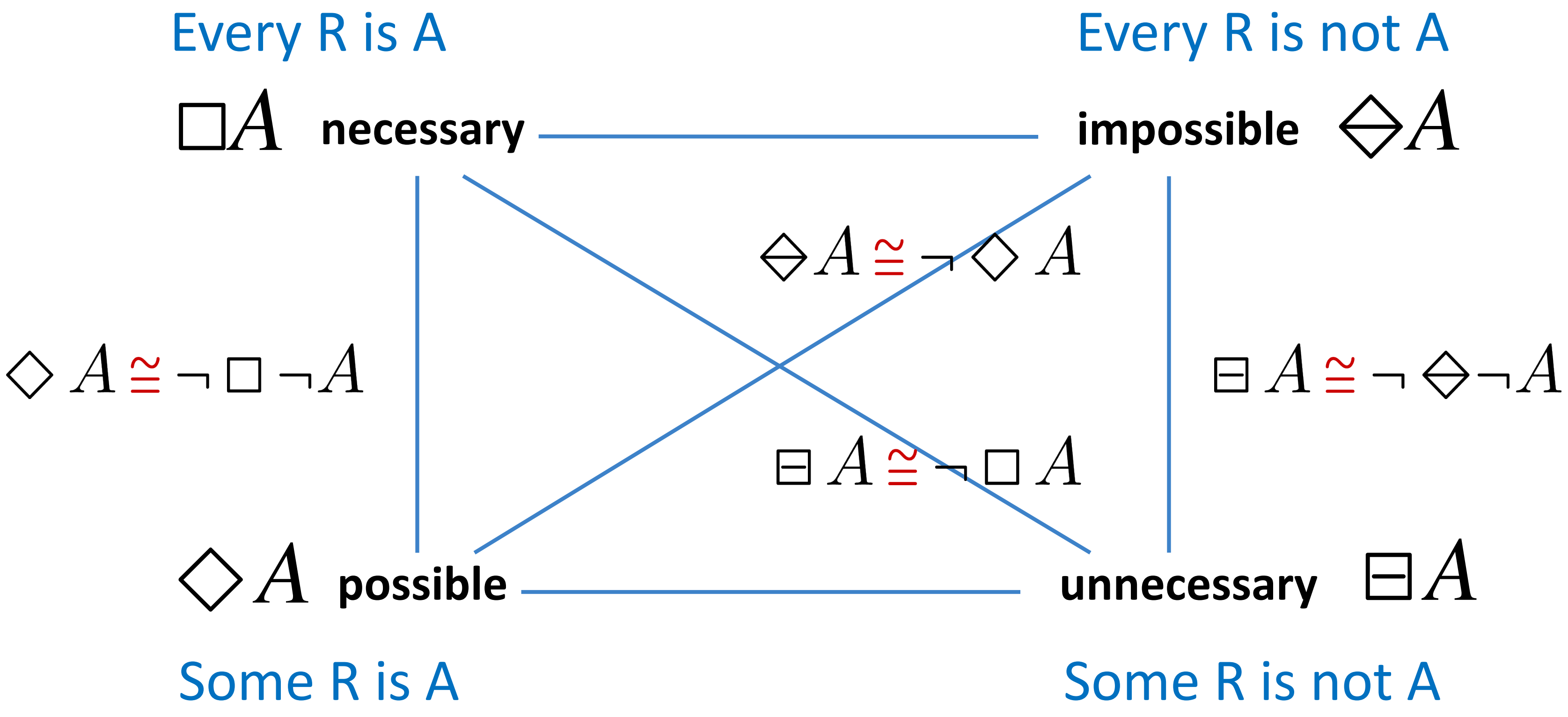
# The Modal Square of Opposition

| | Quality: affirmative | Quality: negative |
|---|---|---|
| Quantity: universal | $\Box A$ **necessary** | **impossible** $\Diamond\!\!\!\!\!-\; A$ |
| Quantity: particular | $\Diamond A$ **possible** | **unnecessary** $\boxminus A$ |

# The Classical (Aristotelian) Square of Opposition

Classically, the modalities are interdefinable with each other via negation

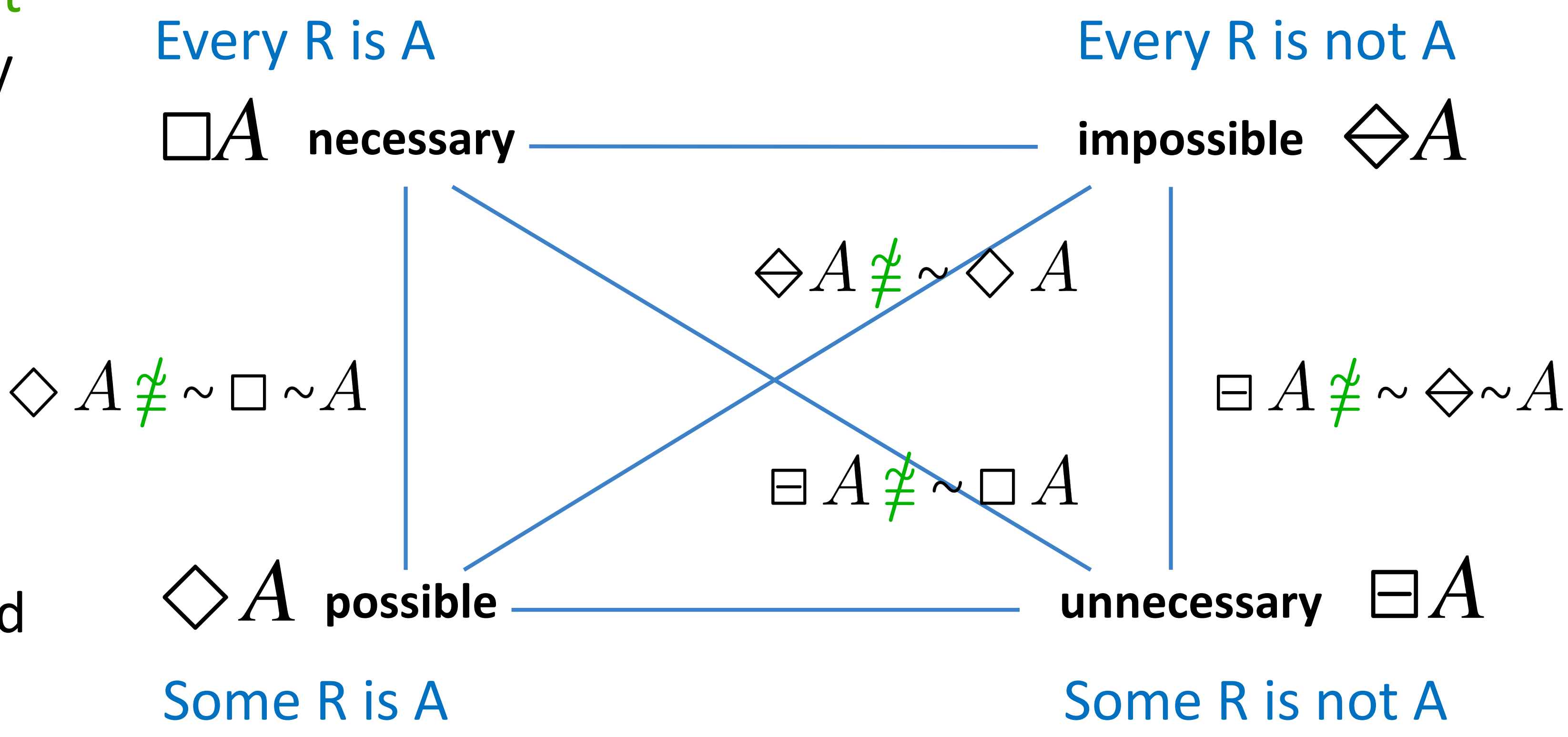
R = reachable possible worlds



# The Constructive Square of Opposition ?

In non-classical logics, the modal quantifiers are not interdefinable any more

R = reachable possible worlds




All modalities need to be treated at a par …

# Constructive (Non-classical) Modalities

Who is going to wash the dishes ?

| necessarily one of us | not possibly me | not possibly me, either | but possibly something happens |
|---|---|---|---|
| $\Box(\mathit{you} \vee \mathit{me})$ | $\sim\Diamond\,\mathit{me}$ | $\sim\Diamond\,\mathit{you}$ | $\Diamond\,\top$ |

constructively, this is perfectly consistent

$\Box\sim\mathit{me}$ $\quad$ $\Box\sim\mathit{you}$

$\Box\sim(\mathit{you} \vee \mathit{me})$

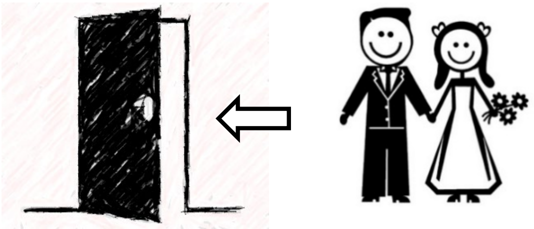
$\Box\,\bot$

$\bot$

classically, such situations cannot exist

# Constructive (Non-classical) Modalities

Who is going to leave through the door first ?



| impossibly both of us | not unne-cessarily me | not unnecessarily me, either | but possibly something happens |
|---|---|---|---|
| $\ominus(you \wedge me)$ | $\sim \boxminus\, me$ | $\sim \boxminus\, you$ | $\Diamond \top$ |

constructively, this is perfectly consistent

$\sim \Diamond(you \wedge me)$ $\quad \Box\, me$ $\quad \Box\, you$

$\Box(you \wedge me)$

$\Diamond(you \wedge me)$

$\bot$

classically, such situations cannot exist

# 2 SYNTAX & INTUITIONISTIC SEMANTICS

# Propositional Language of Modal Logic CKD

- CKD-formulas $\mathcal{F}$ over variables

$$A, B ::= p \mid A \wedge B \mid A \vee B \mid A \to B \mid \Box A \mid \Diamond A \mid \diamondminus A \mid \boxminus A$$

where p ∈ Var = {p, q, …} is a denumerable set of propositional variables.

- Abbreviations

$$\top \equiv p \to p \qquad \bot \equiv \boxminus \top \qquad {\sim}A \equiv A \to \bot \qquad A \leftrightarrow B \equiv (A \to B) \wedge (B \to A)$$

- Restricted Language For $M \subseteq \{\Box, \Diamond, \diamondminus, \boxminus\}$ consider the formulas $\mathcal{F}_M$ in the language $\mathcal{L}_M = \{\bot, \top, \wedge, \vee, \to, \leftrightarrow\} \cup M$ using only modalities from M

# Constructive Modal Theories

- A CKD theory $\mathcal{T}$ in $\mathcal{L}_M$ is a subset of formulas $\mathcal{T} \subseteq \mathcal{F}_M$ closed under
  - deduction (Modus Ponens): If $\mathrm{A} \in \mathcal{T}$ and $\mathrm{A} \to \mathrm{B} \in \mathcal{T}$ then $\mathrm{B} \in \mathcal{T}$
  - uniform substitution: If $\mathrm{A} \in \mathcal{T}$ then $\mathrm{A}\{\mathrm{p} \coloneqq \mathrm{B}\} \in \mathcal{T}$ for $\mathrm{B} \in \mathcal{F}_M$

- A CKD theory $\mathcal{T}$ is constructive if it has the
  - Disjunction Property: If $\mathrm{A} \vee \mathrm{B} \in \mathcal{T}$ then $\mathrm{A} \in \mathcal{T}$ or $\mathrm{B} \in \mathcal{T}$

Recall:

In classical logic $\mathcal{CL}$ we have Excluded Middle and so $\mathrm{p} \vee \neg \mathrm{p} \in \mathcal{CL}$

but $\mathrm{p} \notin \mathcal{CL}$ and $\neg \mathrm{p} \notin \mathcal{CL}$ for all propositions p.

# Intuitionistic Propositional Logic (IPL)

- Axioms

$$A \to (B \to A)$$
$$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$$
$$(A \wedge B) \to A$$
$$(A \wedge B) \to B$$
$$A \to B \to (A \wedge B)$$
$$A \to (A \vee B)$$
$$B \to (A \vee B)$$
$$(A \to C) \to (B \to C) \to ((A \vee B) \to C)$$
$$\bot \to A$$

- Hilbert Deduction

Let $\Gamma, A \subseteq \mathcal{F}_M$ be arbitrary and $\Gamma \vdash A$ given by

$$\frac{A \text{ axiom}}{\Gamma \vdash A}\ \text{ax}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}\ \text{MP}$$

- IPL Theory IPL $=_{df} \{A \in \mathcal{F}_\varnothing \mid \varnothing \vdash A\}$
- Theorem IPL is a constructive theory, i.e., if $\varnothing \vdash A \vee B$ then $\varnothing \vdash A$ or $\varnothing \vdash B$.

# Intuitionistic Kripke Semantics for IPL

- I-frame $\mathfrak{F} = (S, \sqsubseteq)$
  - $S$ non-empty set of states
  - $\sqsubseteq$ intuitionistic accessibility relation, satisfying the frame conditions
    transitive: If $s_1 \sqsubseteq s_2 \sqsubseteq s_3$ then $s_1 \sqsubseteq s_3$
    weakly reflexive: If $s_1 \sqsubseteq s_2 \sqsubseteq s_2$ then $s_1 \sqsubseteq s_1$

- I-model $\mathfrak{M} = (\mathfrak{F}, V)$ consist of a frame $\mathfrak{F} = (S, \sqsubseteq)$ and a
  - valuation function $V \in \mathrm{Var} \to 2^S$

# Intuitionistic Kripke Semantics for IPL

- Satisfaction

$$\mathfrak{M}, s \models \top$$

$$\mathfrak{M}, s \models \bot \text{ iff } s \in F =_{df} \{s \in S \mid s \not\models s\} \text{(fallible states)}$$

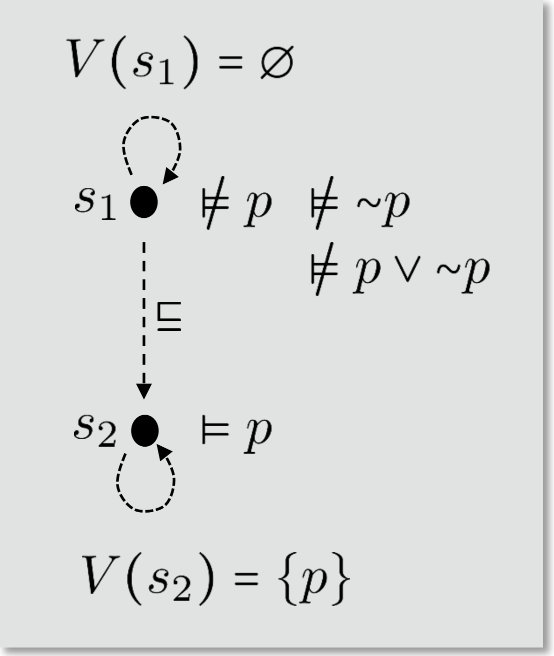
$$\mathfrak{M}, s \models p \text{ iff for all } s' \sqsupseteq s,\ s' \in F \text{ or } p \in V(s')$$

$$\mathfrak{M}, s \models A \wedge B \text{ iff } \mathfrak{M}, s \models A \text{ and } \mathfrak{M}, s \models B$$

$$\mathfrak{M}, s \models A \vee B \text{ iff } \mathfrak{M}, s \models A \text{ or } \mathfrak{M}, s \models B$$

$$\mathfrak{M}, s \models A \to B \text{ iff for all } s' \sqsupseteq s, \text{ if } \mathfrak{M}, s' \models A \text{ then } \mathfrak{M}, s' \models B.$$

$$\mathfrak{M}, s \models {\sim}A \text{ iff for all } s' \sqsupseteq s,\ s' \in F \text{ or } \mathfrak{M}, s' \not\models A$$




- Hereditary Truth If $\mathfrak{M}, s \models A$ and $s \sqsubseteq s'$, then $\mathfrak{M}, s' \models A$
- Validity If $\mathfrak{F} \models A$ iff $\mathfrak{M}, s \models A$ for all $\mathfrak{M} = (\mathfrak{F}, V)$ and $s \in S$
- Soundness & Completeness $\varnothing \vdash A$ iff $\mathfrak{F} \models A$ for all I-frames $\mathfrak{F}$.

# 3 CONSTRUCTIVE MODALITIES

# Intuitionistic/Constructive Modal Logics (incomplete list)

Almost all work on $\Box$ and $\Diamond$ only, very little on $\boxminus$ and ⟠:

- Fitch 1948, Curry 1952, Prior & Bull 1957
- Sotirov 1977, Ono 1977, Fischer-Servi 1981, Vakarelov 1981, Došen 1984, Božić & Došen 1984, Font 1986, Fine 1987
- Plotkin & Stirling 1986, Wijesekera 1990, Masini 1993, Simpson 1994
- Biermann & DePaiva 2000, Bellin & DePaiva & Ritter 2001, Mendler & DePaiva 2005, Mendler & Scheele 2008

# Intuitionistic Modal Logics

- Došen 1984

HK$\Box$ = IPL + ...
in the language $\mathcal{L}_{\Box}$.

$$D\Box 1 =_{df} (\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$$
$$D\Box 2 =_{df} \top \rightarrow \Box \top$$

$$\frac{A \rightarrow B}{\Box A \rightarrow \Box B}\ R\Box$$

HK$\Diamond$ = IPL + ...
in the language $\mathcal{L}_{\Diamond}$.

$$D\Diamond 1 =_{df} \Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$$
$$D\Diamond 2 =_{df} \Diamond \bot \rightarrow \bot$$

$$\frac{A \rightarrow B}{\Diamond A \rightarrow \Diamond B}\ R\Diamond$$

HK$\boxminus$ = IPL + ...
in the language $\mathcal{L}_{\boxminus}$.

$$D\boxminus 1 =_{df} \boxminus(A \wedge B) \rightarrow (\boxminus A \vee \boxminus B)$$
$$D\boxminus 2 =_{df} \boxminus \top \rightarrow \bot$$

$$\frac{A \rightarrow B}{\boxminus B \rightarrow \boxminus A}\ R\boxminus$$

HK$\diamondminus$ = IPL + ...
in the language $\mathcal{L}_{\diamondminus}$.

$$D\diamondminus 1 =_{df} (\diamondminus A \wedge \diamondminus B) \rightarrow \diamondminus(A \vee B)$$
$$D\diamondminus 2 =_{df} \top \rightarrow \diamondminus \bot$$

$$\frac{A \rightarrow B}{\diamondminus B \rightarrow \diamondminus A}\ R\diamondminus$$

# Intuitionistic Modal Logics

Božić & Došen study the Kripke model theory of each HK$\otimes$ independently…

- Proposition [Božić & D, Došen'84]: HK$\otimes$ is constructive for each $\otimes \in \{\Box, \Diamond, \boxminus, ⟐\}$

What about combining the modalities into a single system?

- Fischer-Servi 1980, Plotkin & Stirling 1986, Simpson 1994 constructive

  FS/IK = HK$\Box$ + HK$\Diamond$ + . . . FS5 $=_{df} (\Diamond A \to \Box B) \to \Box(A \to B)$ FS6 $=_{df} \Diamond(A \to B) \to \Box A \to \Diamond B$
  in the language $\mathcal{L}_{\Box\Diamond}$

- Božić & Došen 1984 not constructive

  HK$\Box\Diamond$ = HK$\Box$ + HK$\Diamond$ + . . . $\Box\Diamond 1 =_{df} \Diamond A \vee \Box{\sim}A$ $\Box\Diamond 2 =_{df} {\sim}(\Diamond A \wedge \Box{\sim}A)$
  in the language $\mathcal{L}_{\Box\Diamond}$.

# Combining Positive and Negative Modalities: Došen Theories

- **Definition [Došen theory]** A theory in the full language $\mathcal{L}_{\Box\Diamond\diamondminus\boxminus}$ is a Došen theory if it contains each HK⊗ and is closed under the Regularity Rules R⊗

The smallest Došen theory is the proof-theoretic fusion

$$\mathrm{HK}\Box\Diamond\diamondminus\boxminus = \mathrm{HK}\Box + \mathrm{HK}\Diamond + \mathrm{HK}\diamondminus + \mathrm{HK}\boxminus$$ *„HK-all“*

- **Observation:** The fusion $HK\Box\Diamond\diamondminus\boxminus$ is constructive.

However, $HK\Box\Diamond\diamondminus\boxminus$ has no interaction between the modalities, e.g.

$$\mathrm{HK}\Box\Diamond\diamondminus\boxminus \nvdash (\Box A \wedge \Diamond B) \rightarrow \Diamond(A \wedge B) \qquad \mathrm{HK}\Box\Diamond\diamondminus\boxminus \nvdash {\sim}(\Diamond A \wedge \diamondminus A)$$

Interaction comes from a frame-theoretic fusion of the HK⊗ ...

# Došen-style (bi-relational) Interpretation of Modalities (HK$\otimes$)

The HK$\otimes$ semantics extend intuitionistic Kripke-style frame semantics for IPL

- C-frames $\mathfrak{F} = (S, \sqsubseteq, R)$
  - $(S, \sqsubseteq)$ is an I-frame
  - *R* modal accessibility relation
- C-models $\mathfrak{M} = (\mathfrak{F}, V)$ are C-frames $\mathfrak{F} = (S, \sqsubseteq, R)$ plus valuation $V \in \mathsf{Var} \to 2^S$
- A HK$\otimes$-frame is a C-frame $\mathfrak{F}$ in which $\sqsubseteq$ is reflexive and R satisfies the F$\otimes$ frame condition (see below)
- A HK$\otimes$-model is a C-model $\mathfrak{M} = (\mathfrak{F}, V)$ in which $\mathfrak{F}$ is a HK$\otimes$-frame.

# Došen-style (bi-relational) Interpretation of Modalities (HK$\otimes$)

- Modal Truth Clauses

$$\mathfrak{M}, s \Vdash \Box A \text{ iff } \forall x.\, s\, R\, x \;\Rightarrow\; \mathfrak{M}, x \vDash A$$

$$\mathfrak{M}, s \Vdash \ominus A \text{ iff } \forall x.\, s\, R\, x \;\Rightarrow\; \mathfrak{M}, x \nvDash A$$

$$\mathfrak{M}, s \Vdash \Diamond A \text{ iff } \exists x.\, s\, R\, x \,\&\, \mathfrak{M}, x \vDash A$$

$$\mathfrak{M}, s \Vdash \boxminus A \text{ iff } \exists x.\, s\, R\, x \,\&\, \mathfrak{M}, x \nvDash A$$

- Frame Properties

$$F\Box =_{df} (\sqsubseteq ; R) \subseteq (R ; \sqsubseteq)$$

$$F\ominus =_{df} (\sqsubseteq ; R) \subseteq (R ; \sqsupseteq)$$

$$F\Diamond =_{df} (\sqsupseteq ; R) \subseteq (R ; \sqsupseteq)$$

$$F\boxminus =_{df} (\sqsupseteq ; R) \subseteq (R ; \sqsubseteq)$$

- Hereditary Truth If $s \sqsubseteq s'$ and $\mathfrak{M}, s \Vdash A$ then $\mathfrak{M}, s' \Vdash A$

- Soundness & Completeness [Došen'84]

For $A \in \mathcal{F}_{\otimes}$, HK$\otimes$ $\vdash A$ iff $\mathfrak{F} \Vdash A$ for all HK$\otimes$-frames $\mathfrak{F}$

# Combining Modalities: Došen Frames

Let $\mathcal{DF}$ be the C-frames satisfying all F$\otimes$ frame properties simultaneously and

$$\mathrm{DF} = \{A \mid \forall \mathfrak{F} \in \mathcal{DF}.\, \mathfrak{F} \models A\}$$

the theory induced by the $\mathcal{DF}$ frames.

- Observation: DF is a Došen theory

Now we have useful modal interaction

$$\mathrm{DF} \vdash (\Box A \wedge \Diamond B) \rightarrow \Diamond(A \wedge B)$$

$$\mathrm{DF} \vdash {\sim}(\Diamond A \wedge \diamondminus A) \qquad \mathrm{DF} \vdash {\sim}(\Box A \wedge \boxminus A)$$

Trouble is, there is too much modal interaction in DF

# Došen Frames: Too Much Modal Interaction

On Došen frames no modality carries constructive content …

- Proposition

  Let $\oplus \in \{\Box, \Diamond, \ominus\!\!\!\!\Diamond, \boxminus\}$ be a modality and $\ominus$ its associated "contradictory" partner.

  $$\mathrm{DF} \vdash \oplus A \leftrightarrow {\sim} \ominus A \qquad\qquad \mathrm{DF} \vdash \oplus A \vee {\sim} \oplus A$$

- Corollary* The Došen theory DF is not constructive !

In CKD, we introduce a new semantic interpretation of modalities …

*[We generalise an observation made by S. A. Drobyshevich for the logic N*]

# CKD Semantics: Forcing Heredity without Frame Conditions

- Force heredity without any frame conditions
- Use doubly quantified* (constructive) interpretation
- $\sqsubseteq$ need not be reflexive (fallible worlds)

$$\mathfrak{M}, s \vDash \Diamond A \;\Leftrightarrow\; \forall s' \sqsupseteq s.\, \exists x.\, (s' R x \;\&\; \mathfrak{M}, x \vDash A)$$

$$\mathfrak{M}, s \vDash \Box A \;\Leftrightarrow\; \forall s' \sqsupseteq s.\, \forall x.\, (s' R x \Rightarrow \mathfrak{M}, x \vDash A)$$

$$\mathfrak{M}, s \vDash \ominus\!\!\!\Diamond A \;\Leftrightarrow\; \forall s' \sqsupseteq s.\, \forall x.\, (s' R x \Rightarrow \mathfrak{M}, x \nvDash A)$$

$$\mathfrak{M}, s \vDash \boxminus A \;\Leftrightarrow\; \forall s' \sqsupseteq s.\, \exists x.\, (s' R x \;\&\; \mathfrak{M}, x \nvDash A)$$

$\vDash$ refines Dosen's $\Vdash$ for constructive interpretation

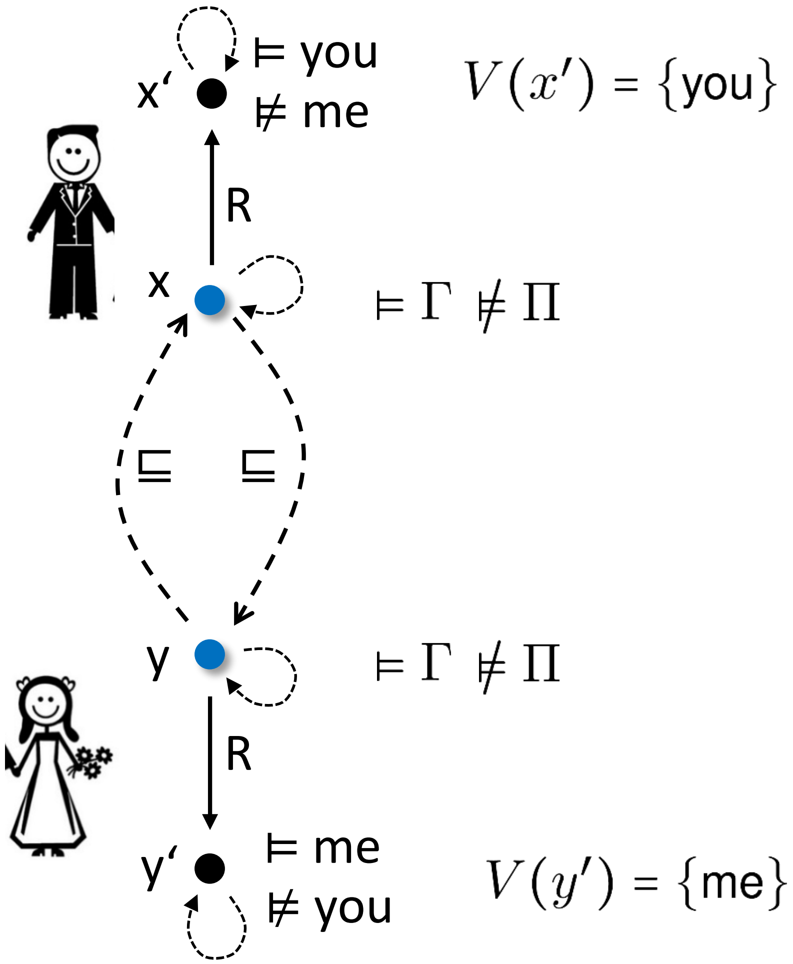
*Notes

- for $\Box$ originally by Plotkin & Stirling 1986
- for $\Diamond$ originally by Wijesekera 1990, Fairtlough & Mendler 1994
- distinguishes `constructive‘ from `intuitionistic‘ modal logics
- is extended here for the negative modalities $\boxminus$, $\ominus\!\!\!\Diamond$

# CKD Admits Classically Inconsistent (Metastable) States

$$
\begin{aligned}
\Gamma \;=\; \{ & \Box(\text{you} \vee \text{me}), \diamondminus(\text{you} \wedge \text{me}), \\
& \sim \Diamond\, \text{me}, \sim \Diamond\, \text{you}, \\
& \sim \diamondminus \text{me}, \sim \diamondminus \text{you}, \\
& \sim \Box\, \text{me}, \sim \Box\, \text{you}, \\
& \sim \boxminus\, \text{me}, \sim \boxminus\, \text{you}, \\
& \Diamond \top \}
\end{aligned}
$$

$$\Pi = \{\Diamond\text{me}, \diamondminus\text{me}, \Diamond\text{you}, \diamondminus\text{you}, \Box\text{me}, \boxminus\text{me}, \Box\text{you}, \boxminus\text{you}\}$$



# 4 HILBERT DEDUCTION (HCKD)

# Hilbert Axiomatisation of CKD = IPL + …

- **K-axioms**

$\Box K =_{df} \Box(A \to B) \to \Box A \to \Box B$ $\quad$ $\ominus K =_{df} \Box(A \to B) \to \ominus B \to \ominus A$

$\Diamond K =_{df} \Box(A \to B) \to \Diamond A \to \Diamond B$ $\quad$ $\boxminus K =_{df} \Box(A \to B) \to \boxminus B \to \boxminus A$

# Hilbert Axiomatisation of CKD = IPL + …

- **K-axio**

  $\Box K =$

  $\Diamond K =$

„2-axioms“ capture the interaction
between impossibility and every other modality

- **2-axioms**

$$\Box 2 =_{df} \diamondminus A \to \Box(A \vee B) \to \Box B \qquad \diamondminus 2 =_{df} \diamondminus A \to \diamondminus B \to \diamondminus(A \vee B)$$

$$\Diamond 2 =_{df} \diamondminus A \to \Diamond(A \vee B) \to \Diamond B \qquad \boxminus 2 =_{df} \diamondminus A \to \boxminus B \to \boxminus(A \vee B)$$

# Hilbert Axiomatisation of CKD = IPL + …

- **K-axioms**

$\Box K =_{df}$

$\Diamond K =_{df}$

> If a disjunction $A \vee B$ is necessary & disjunct $A$ is impossible
>
> then the other disjunct $B$ is necessary

- **2-axioms**

$\Box 2 =_{df} ⟠ A \to \Box(A \vee B) \to \Box B$ $\qquad$ $⟠ 2 =_{df} ⟠ A \to ⟠ B \to ⟠(A \vee B)$

$\Diamond 2 =_{df} ⟠ A \to \Diamond(A \vee B) \to \Diamond B$ $\qquad$ $\boxminus 2 =_{df} ⟠ A \to \boxminus B \to \boxminus(A \vee B)$

# Hilbert Axiomatisation of CKD = IPL + …

- **K-axioms**

$\Box K =_{df} \Box(A \to B) \to \Box A \to \Box B \qquad \diamondminus K =_{df} \Box(A \to B) \to \diamondminus B \to \diamondminus A$

$\Diamond K =_{df} \Box(A \to B) \to \Diamond A \to \Diamond B \qquad \boxminus K =_{df} \Box(A \to B) \to \boxminus B \to \boxminus A$

„N-axioms“ capture the interaction between 3 different modalities …

- **2-ax**

$\Box 2$

$\Diamond 2 =_{df} \diamondminus A \to$ … $\Diamond B \qquad \boxminus 2 =_{df} \diamondminus A \to \boxminus B \to \boxminus(A \vee B)$

- **N-axioms** $N5 =_{df} \diamondminus(A \wedge B) \to \Diamond A \to \boxminus B \qquad N6 =_{df} \Box(A \vee B) \to \boxminus A \to \Diamond B$

# Hilbert Axiomatisation of CKD = IPL + …

- **K-axioms**

$\Box K =_{df} \Box(A \to B) \to \Box A \to \Box B$ $\quad$ $\diamondminus K =_{df} \Box(A \to B) \to \diamondminus B \to \diamondminus A$

$\Diamond K =_{df} \Box(A \to B) \to \Diamond A \to \Diamond B$ $\quad$ $\boxminus K =_{df} \Box(A \to B) \to \boxminus B \to \boxminus A$

If a conjunction $A \wedge B$ is impossible and conjunct $A$ is possible

then the other conjunct $B$ is unnecessary

- **2-ax**

$\Box 2$

$\Diamond 2 =_{df} \diamondminus A \to \Diamond(A$ $\quad$ $\boxminus 2 =_{df} \diamondminus A \to \boxminus B \to \boxminus(A \vee B)$

- **N-axioms** $N5 =_{df} \diamondminus(A \wedge B) \to \Diamond A \to \boxminus B$ $\quad$ $N6 =_{df} \Box(A \vee B) \to \boxminus A \to \Diamond B$

- **Rules**

$$\frac{A \quad A \to B}{B}\,\text{MP} \qquad \frac{A}{\Box B}\,\text{Nec}$$

# Hilbert Axiomatisation of CKD = IPL + …

- **K-axioms**

$$\Box K =_{df} \Box(A \to B) \to \Box A \to \Box B \qquad \diamondminus K =_{df} \Box(A \to B) \to \diamondminus B \to \diamondminus A$$

$$\Diamond K =_{df} \Box(A \to B) \to \Diamond A \to \Diamond B \qquad \boxminus K =_{df} \Box(A \to B) \to \boxminus B \to \boxminus A$$

- **2-axioms**

$$\Box 2 =_{df} \diamondminus A \to \Box(A \vee B) \to \Box B \qquad \diamondminus 2 =_{df} \diamondminus A \to \diamondminus B \to \diamondminus(A \vee B)$$

$$\Diamond 2 =_{df} \diamondminus A \to \Diamond(A \vee B) \to \Diamond B \qquad \boxminus 2 =_{df} \diamondminus A \to \boxminus B \to \boxminus(A \vee B)$$

- **N-axioms** $N5 =_{df} \diamondminus(A \wedge B) \to \Diamond A \to \boxminus B$ $\quad N6 =_{df} \Box(A \vee B) \to \boxminus A \to \Diamond B$

- **Rules**

$$\frac{A \quad A \to B}{B}\ \text{MP} \qquad \frac{A}{\Box B}\ \text{Nec}$$

# CKD = Conservative Core for Modal Theories

| Theory Fragment | Logic | |
|---|---|---|
| $\mathcal{L}_{\Box\Diamond}(\mathrm{CKD})$ | CK | Mendler&DePaiva'05, Mendler&Scheele'10 |
| $\mathcal{L}_{\wedge,\vee,\rightarrow,\ominus}(\mathrm{CKD})$ | taking $\ominus$ as $\neg$ N | Došen'86 |
| $\mathcal{L}_{\Box}(\mathrm{CKD})$ | HK$\Box$ | Došen'84 |

# CKD = Conservative Core for Modal Theories

| Theory Fragment | „aka“ Name | |
|---|---|---|
| $\mathcal{L}_{\Box\Diamond}(\mathrm{CKD} + \mathrm{D}\Diamond 2)$ | $\mathrm{IPL} + \Box K + \Diamond K + Nec$ | Wijesekera’90 |
| $\mathcal{L}_{\Diamond}(\mathrm{CKD} + \{\mathrm{D}\Diamond 1, \mathrm{D}\Diamond 2\})$ | $\mathrm{HK}\Diamond$ | Došen’84 |
| $\mathcal{L}_{\Box\Diamond}(\mathrm{CKD} + \{\mathrm{D}\Diamond 1, \mathrm{D}\Diamond 2, \mathrm{IK5}\})$ | IK | Fischer-Servi’81, Plotkin & Stirling’86, Simpson’94 |
| $\mathcal{L}_{\Box\Diamond}(\mathrm{CKD} + \{\Box\Diamond 1, \Box\Diamond 2\})$ | not constructive $\mathrm{HK}\Box\Diamond$ | Bosic & Došen’84 |

$\mathrm{D}\Diamond 1 = \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$

$\mathrm{D}\Diamond 2 = {\sim}\Diamond\bot$

$\mathrm{IK5} = (\Diamond A \to \Box B) \to \Box(A \to B)$

$\Box\Diamond 1 = \Diamond A \vee \Box {\sim} A$

$\Box\Diamond 2 = {\sim}(\Diamond A \vee \Box {\sim} A)$

# CKD = Conservative Core for Modal Theories

| Theory Fragment | „aka“ Name | |
|---|---|---|
| $\mathcal{L}_{\ominus}(\mathsf{CKD} + \mathsf{D}\ominus 2)$ | HK$\ominus$ | Došen’84 |
| $\mathcal{L}_{\boxminus}(\mathsf{CKD} + \mathsf{D}\boxminus 1)$ | HK$\boxminus$ | Došen’84 |
| $\mathcal{L}_{\wedge,\vee,\to,\ominus}(\mathsf{CKD} + \{\mathsf{D}\ominus 2, \mathsf{N}\ominus 1, \mathsf{N}\ominus 2\})$ | not constructive taking $\ominus$ as $\neg$ — $N^*$ | Cabalar, Odintsov, Pearce’06 |
| $\mathcal{L}_{\wedge,\vee,\to,\boxminus}(\mathsf{CKD} + \{\mathsf{D}\boxminus 1, \mathsf{N}\boxminus 1, \mathsf{N}\boxminus 2\})$ | not constructive taking $\boxminus$ as $\neg$ — $N^*$ | Cabalar, Odintsov, Pearce’06 |

$\mathsf{D}\boxminus 1 = \boxminus(A \wedge B) \to (\boxminus A \vee \boxminus B)$

$\mathsf{N}\boxminus 1 =_{df} (\boxminus A \wedge \boxminus B) \to \boxminus(A \vee B)$

$\mathsf{D}\ominus 2 = \ominus\bot$

$\mathsf{N}\boxminus 2 =_{df} \boxminus\bot$

$\mathsf{N}\ominus 1 =_{df} \ominus(A \wedge B) \to (\ominus A \vee \ominus B)$

$\mathsf{N}\ominus 2 =_{df} \sim\ominus\top$

# What about Došen Theories?

| Theory Fragment | „aka“ Name | |
|---|---|---|
| $\mathsf{CKD} + \{D\boxminus 1, D\Diamond 2, D\Diamond 2, D⟠ 2\}$ | Constructive Došen Theory CDT | New (M & S & B) |

- Proposition The axiomatic theory

$$\mathrm{CDT} =_{df} \mathrm{CKD} + \{D\Box 1, D\Box 2, D\Diamond 1, D\Diamond 2, D\boxminus 1, D\boxminus 2, D⟠ 1, D⟠ 2\}$$
$$= \mathrm{CKD} + \{D\Diamond 1, D\Diamond 2, D\boxminus 1, D⟠ 2\}$$

is a constructive Došen theory (extending $\mathsf{HK}\Box\Diamond⟠\boxminus$ )

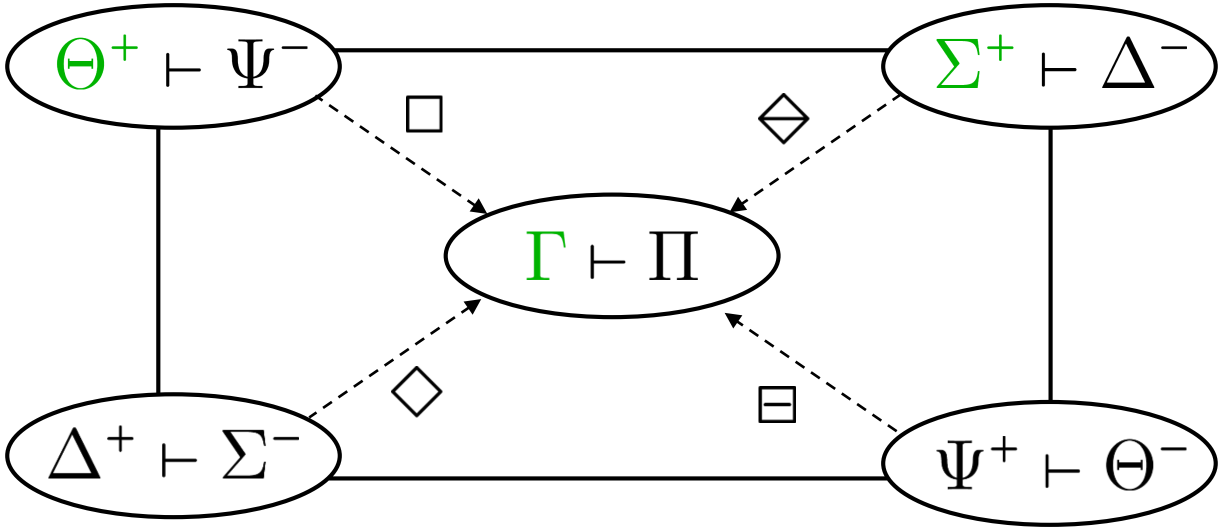
- Proposition

The extensions $\mathsf{CDT} + \Diamond\top$ and $\mathsf{CDT} + \boxminus\bot$ are not constructive

# 5 GENTZEN SEQUENT CALCULUS (GCKD)

# Gentzen-style Sequent Calculus GCDK: The Došen Square

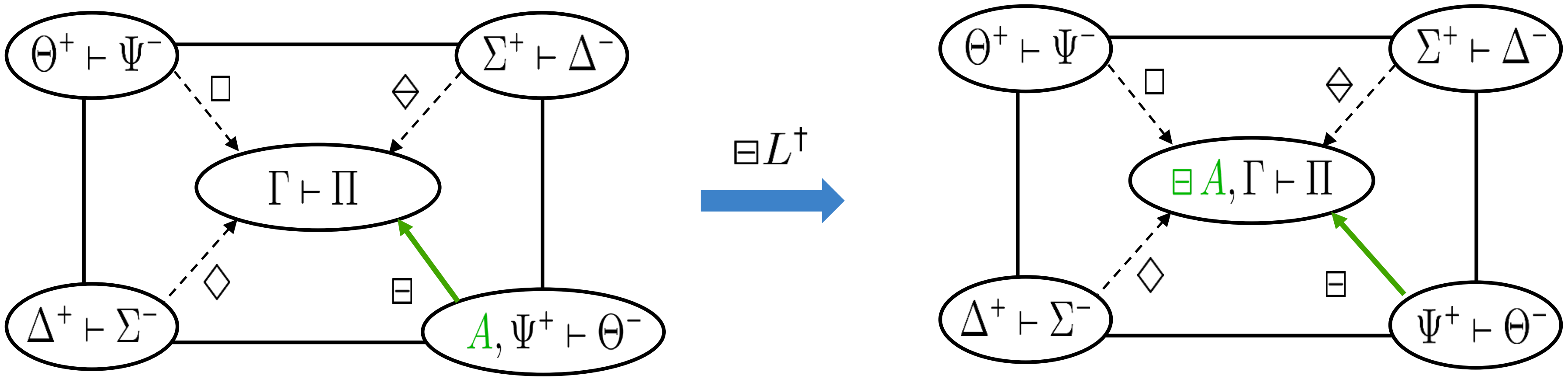
- GCDK sequents are structures $\Gamma * \Delta * \Theta \vdash \Pi * \Sigma * \Psi$ where $\Gamma, \Delta, \Theta, \Pi, \Sigma, \Psi$ are finite (possibly empty) sets of formulas.
- Each of the sets $X \in \{\Delta, \Theta, \Sigma, \Psi\}$ contains +/- signed formulas $X^+, X^-$
- A sequent provides a formalisation of the Square of Opposition as follows:



# The Došen Square: Left Introduction of $\boxminus$ from South East

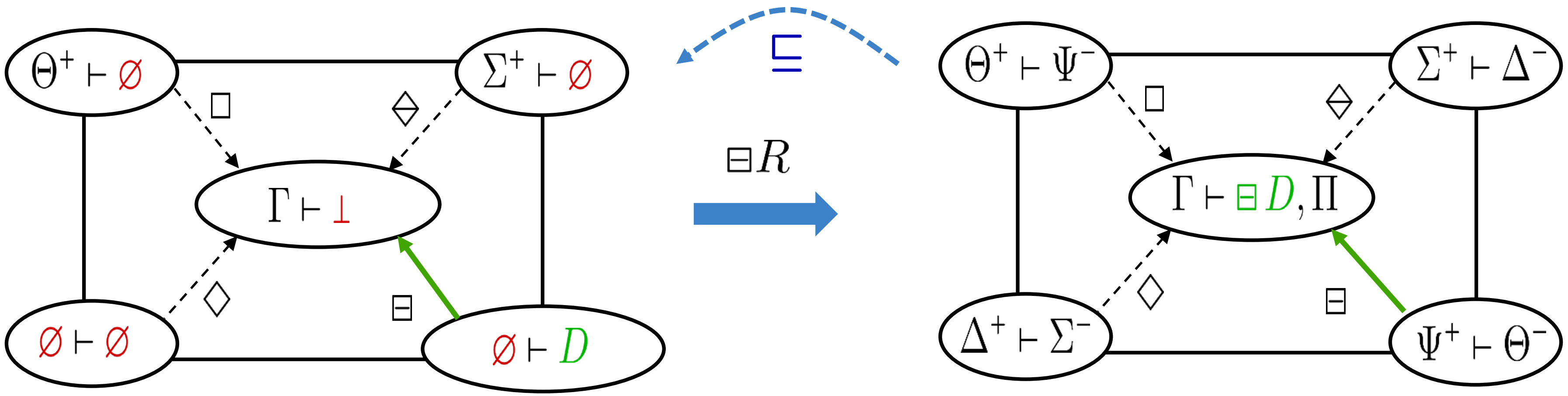
$$\frac{\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star A^+, \Psi}{\boxminus A, \Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\ \boxminus L^\dagger$$

(†) strictness side condition: $|\Delta \cup \Pi \cup \Psi| \geq 1$



# The Došen Square: Right Introduction of ⊟ from South East

$$\frac{\Gamma \star \varnothing \star D^-, \Theta^+ \vdash \bot \star \Sigma^+ \star \varnothing}{\Gamma \star \Delta \star \Theta \vdash \boxminus D, \Pi \star \Sigma \star \Psi} \boxminus R$$




⊟R corresponds to an intuitionistic step, thus some corners must be cleared

# The Došen Square: Grand Modal Dispatch

- In forward direction, the cp*-rules introduce polarity signs for formulas from $\Gamma, \Pi$
- In backwards direction, they realise a modal step

$$\frac{B, \Theta^{+} \star \varnothing \star \varnothing \vdash \Sigma^{+} \star \varnothing \star \varnothing}{\Gamma \star B^{-}, \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\, cpL^{-} \qquad \frac{\Theta^{+} \star \varnothing \star \varnothing \vdash F, \Sigma^{+} \star \varnothing \star \varnothing}{\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star F^{-}, \Psi}\, cpR^{-}$$

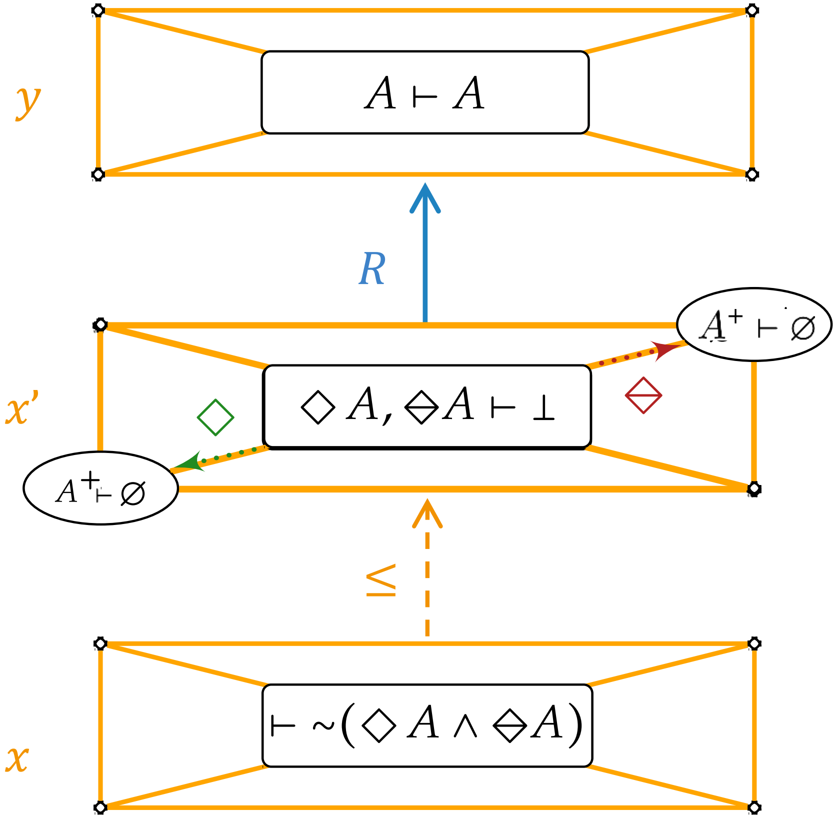
$$\frac{B, \Theta \star \varnothing \star \varnothing \vdash \Sigma \star \varnothing \star \varnothing}{\Gamma \star B^{+}, \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi}\, cpL^{+} \qquad \frac{\Theta \star \varnothing \star \varnothing \vdash F, \Sigma \star \varnothing \star \varnothing}{\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star F^{+}, \Psi}\, cpR^{+}$$

Inspired by cp = contraposition rule of N*

# The Došen Square: Example derivation

- Proof of incompatibility of contradictories possible and impossible, i.e., $\sim(\Diamond A \wedge \diamondminus A)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \star \varnothing \star \varnothing \vdash A \star \varnothing \star \varnothing}\,Ax}{\varnothing \star A^{+} \star \varnothing \vdash \bot \star A^{+} \star \varnothing}\,cpL^{+}}{\Diamond A, \diamondminus A \star \varnothing \star \varnothing \vdash \bot \star \varnothing \star \varnothing}\,\diamondminus L\ \Diamond L}{\Diamond A \wedge \diamondminus A \star \varnothing \star \varnothing \vdash \bot \star \varnothing \star \varnothing}\,\wedge L}{\varnothing \star \varnothing \star \varnothing \vdash \sim(\Diamond A \wedge \diamondminus A) \star \varnothing \star \varnothing}\,\sim R$$



# Gentzen-style Sequent Calculus GCKD: Results

- Theorem [Mendler, Scheele, Burke (Tableaux'21)]
  - GCKD is sound and complete for C-models (canonical model via consistent, saturated Došen squares)
  - Structural translation between GCKD and the Hilbert Calculus
    - polarised sequents $\Gamma \star \Delta \star \Theta \vdash \Pi \star \Sigma \star \Psi$ with $|\Theta^- \cup \Sigma^-| \leq 1$ have global meaning as formulas

- Observation
  - GCKD is cut-free and has the sub-formula property $\Rightarrow$ finite search space

# 6 CONCLUSION

# Summary of Results

- Theorem [Mendler, Scheele, Burke (Tableaux‘21)]
  - HCKD is sound and complete for C-frames
  - CKD is constructive (satisfies the Disjunction Property)
  - CKD has finite model property and is decidable
  - Existing modal theories arise as fragments and axiomatic extensions of CKD (see our Tableaux‘21 paper)

- Proposition The axiomatic theory

$$\begin{aligned}\mathrm{CDT} \quad &=_{df} \quad \mathrm{CKD} + \{\mathrm{D}\Box 1, \mathrm{D}\Box 2, \mathrm{D}\Diamond 1, \mathrm{D}\Diamond 2, \mathrm{D}\boxminus 1, \mathrm{D}\boxminus 2, \mathrm{D}\diamondminus 1, \mathrm{D}\diamondminus 2\} \\ &= \quad \mathrm{CKD} + \{\mathrm{D}\Diamond 1, \mathrm{D}\Diamond 2, \mathrm{D}\boxminus 1, \mathrm{D}\diamondminus 2\}\end{aligned}$$

is a constructive Došen theory

# Conclusion & Open Questions

CKD: first constructive modal Došen theory with positive and negative modalities

- CKD/CDT as type system / $\lambda$-calculus (Curry-Howard Correspondence?)
  - syntactic cut-elimination proof
  - Note: a $\lambda$-calculus for CKD in $\mathcal{L}_{\Box,\Diamond}$ exists [Mendler & Scheele, Fundam. Inform. 2014]
- Neighborhood semantics [Kojima 2012, Dalmonte 2020] for CKD
- Terminating ("maximal" duplication-free?) sequent calculi for CKD

Thank you for your attention ! Questions ?