On the bimodality of the distribution of the S&P 500's distortion: empirical evidence and theoretical explanations

Noemi Schmitt and Frank Westerhoff

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Bamberg Economic Research Group
Bamberg University
Feldkirchenstraße 21
D-96052 Bamberg
Telefax: (0951) 863 5547
Telephone: (0951) 863 2687
felix.stuebben@uni-bamberg.de
http://www.uni-bamberg.de/vwl/forschung/berg/

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Redaktion:
Dr. Felix Stübben*

* felix.stuebben@uni-bamberg.de

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Noemi Schmitt and Frank Westerhoff **
University of Bamberg, Department of Economics

Abstract

After showing that the distribution of the S&P 500's distortion, i.e. the log difference between its real stock market index and its real fundamental value, is bimodal, we demonstrate that agent-based financial market models may explain this puzzling observation. Within these models, speculators apply technical and fundamental analysis to predict asset prices. Since destabilizing technical trading dominates the market near the fundamental value, asset prices tend to be either overvalued or undervalued. Interestingly, the bimodality of the distribution of the S&P 500's distortion confirms an implicit prediction of a number of seminal agent-based financial market models.

Keywords

Stock market dynamics; bubbles and crashes; chartists and fundamentalists; nonlinear dynamics; bimodality tests; time series analysis.

JEL classification

G12; G14; G17.

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** Contact: Frank Westerhoff, University of Bamberg, Department of Economics, Feldkirchenstrasse 21, 96045 Bamberg, Germany. Email: frank.westerhoff@uni-bamberg.de. Phone: +49 951 8632634.

1 Introduction

The goal of our paper is twofold. We first present empirical evidence indicating that the distribution of the S&P 500's distortion, i.e. the log difference between its real stock market index and its real fundamental value, is bimodal. While the S&P 500 fluctuates in an intricate manner around its fundamental value, we show that it spends relatively more time in bull and bear markets than in the vicinity of its fundamental value.[1] The distribution of the S&P 500's distortion is thus – contrary to what one would expect – not unimodal but possesses a bimodal shape. We then demonstrate that this puzzling observation may be explained by agent-based financial market models. Since speculators rely within these models on technical and fundamental analysis to predict asset prices, their dynamics depends on two competing forces. As we will see, it is the repeated comeback of destabilizing technical forces near fundamental values that tends to keep markets distorted. We would like to stress that the bimodality of the distribution of the S&P 500's distortion, as documented in our paper, confirms an implicit prediction of a number of seminal agent-based financial market models that, until now, has been largely neglected.

The empirical part of our paper rests on Shiller's (2015) proposal on how to compute the S&P 500's fundamental value. His unique historical dataset from January 1871 to December 2015 gives us access to 1,740 monthly observations of the real S&P 500 and its real dividend payments. In his Nobel Prize Lecture, Shiller (2015) determines the real fundamental value of the S&P 500 by discounting its real dividend payments, assuming a constant real discount rate and a constant real growth rate of the last observed real dividend. We define the S&P 500's distortion as the log difference between the real S&P 500 and its real fundamental value. Visual impression

[1] We follow Day and Huang (1990) and classify a market as a bull (bear) market when prices are above (below) fundamental values.

as well as Silverman's (1981) statistical mode test indicate that the distribution of the S&P 500's distortion is bimodal, i.e. the S&P 500 spends relatively more time in bull and bear markets than in the neighborhood of its fundamental value. In our view, this is very surprising since the distribution of the S&P 500's distortion possesses a local minimum at the very place where one would expect to find a global peak.

As is well known, standard linear time series models do not give rise to such a bimodal distribution. However, in order to rule out the S&P 500's bimodal distributed distortion being due to finite sample effects, assuming that the true distribution is unimodal, we conduct a simple simulation study in which we hypothetically assume that standard linear time series models represent the true data-generating process of the S&P 500's distortion. We can thus compare the magnitude of the dip in the bimodal distribution of the S&P 500's distortion to those one may encounter in simulated distributions derived from such models. Searching within a large class of standard linear time series models, common model selection criteria favor an ARMA (2,2) model as the true data generating process. Although simulated time series resemble the path of the S&P 500's distortion, at least at first sight, our simulation study reveals that the dip we observe empirically is very unlikely to occur in an environment in which the true data-generating process is given by standard linear time series models such as an ARMA (2,2) model. From this perspective, we can furthermore conclude that linear economic dynamic models are unable to explain the bimodality of the distribution of the S&P 500's distortion. Or, in other words, our simulation study suggests that the bimodality of the S&P 500's distribution may be due to nonlinear forces.

Over the last couple of years, agent-based financial market models have improved our understanding of the functioning of financial markets. For surveys of this line of research see, for instance, LeBaron (2006), Chiarella et al. (2009), Hommes and Wagener (2009) and Lux (2009).

Within these models, speculators rely on a nonlinear mix of technical and fundamental analysis to determine their trading behavior. While technical analysis (Murphy 1991) seeks to derive trading signals out of past asset price movements, fundamental analysis (Graham and Dodd 1951) predicts that asset prices revert towards their fundamental values.[2] Agent-based financial market models demonstrate that endogenous interactions between destabilizing technical trading rules and stabilizing fundamental trading rules may give rise to realistic asset price dynamics. Early contributions in this direction include Zeeman (1974), Day and Huang (1990), Kirman (1991), Chiarella (1992), de Grauwe et al. (1993), Lux (1995), Brock and Hommes (1998), LeBaron et al. (1999) and Farmer and Joshi (2002) while more recent approaches include Chiarella et al. (2007), Huang et al. (2010), LeBaron (2012), Anufriev and Hommes (2012), Anufriev and Tuinstra (2013), Schmitt and Westerhoff (2014), He and Li (2015) and He and Zheng (2016).

A number of agent-based financial market models may be used to explain the bimodality of the distribution of the S&P 500's distortion. However, the model by Gaunersdorfer and Hommes (2007) seems to us to be the ideal model for understanding the key mechanism that causes this property. Gaunersdorfer and Hommes (2007) propose a standard discounted value asset pricing model in which speculators can invest in a risk-free asset, paying a fixed rate of return, or in a risky asset, paying an uncertain dividend. Moreover, speculators switch between technical and fundamental analysis rules to predict future asset prices with respect to the rules' past profitability and the market's deviation from its fundamental value. To be precise, speculators prefer rules which have produced higher profits in the recent past and yet, in fear of a bursting bubble, they increasingly opt for fundamental analysis as the market's misalignment increases. Gaunersdorfer and Hommes (2007) show that their calibrated model matches

[2] Laboratory experiments surveyed in Hommes (2011) and questionnaire studies summarized in Menkhoff and Taylor (2007) unanimously confirm that financial market participants rely on technical and fundamental analysis.

important statistical properties of the S&P 500 quite well, including bubbles and crashes, excess volatility, fat-tailed return distributions, uncorrelated returns and volatility clustering.

The deterministic skeleton of the calibrated model by Gaunersdorfer and Hommes (2007) gives rise to a locally stable limit cycle, surrounding a coexisting locally stable fundamental steady state. As it turns out, the bimodality of the distribution of the S&P 500's distortion may be explained by the limit cycle's properties. Close to the fundamental steady state, the dynamics of the model is driven by the trend-extrapolating behavior of chartists. Their trading behavior rapidly pushes the asset price away from its fundamental value. As the market's misalignment increases, fundamental analysis becomes more popular. However, the mean reversion pressure exercised by fundamentalists is rather weak and thus it takes a while for the price to approach its fundamental value. During this adjustment process, both technical and fundamental rules are profitable. However, since the market's misalignment shrinks, more and more speculators return to technical analysis. As a result, the momentum of the adjustment dynamics accelerates and the price overshoots its fundamental value, tracing out a new bubble path. To sum up: fundamental analysis manages to drive asset prices towards fundamental values, but the consequent revival of destabilizing technical rules tends to keep the market distorted. Together, these forces render the distribution of the distortion bimodal. We show that the same mechanism is at work in the calibrated (stochastic) model by Gaunersdorfer and Hommes (2007). It is worth noting how well their model matches the bimodality of the distribution of the S&P 500's distortion, although it was designed with a different purpose in mind.

This outcome does not depend on the details of the model by Gaunersdorfer and Hommes (2007), but can also be observed in a number of related models. For instance, Franke and Westerhoff (2012) propose an agent-based financial market model in which speculators switch between technical and fundamental trading rules with respect to predisposition effects, herding behavior and market misalignments. Franke and Westerhoff (2012) estimate their model using the method of simulated moments, and report that their approach is quite powerful in matching a number of salient statistical properties of the S&P 500. Computing the distribution of the market's distortion for this estimated agent-based model reveals clear signs of bimodality. Since similar results can be detected in many other frameworks – we explicitly explore the seminal contributions by Zeeman (1974), Day and Huang (1990), Chiarella (1992), de Grauwe et al. (1993), Lux (1995) and Brock and Hommes (1998) – one may conclude that the surprising bimodality of the distribution of the S&P 500's distortion may be explained by agent-based financial market models. Put differently, the bimodality of the distribution of the S&P 500's distortion, as documented in our paper, confirms an implicit prediction of many agent-based financial market models.

The rest of our paper is organized as follows. In Section 2, we provide empirical evidence indicating that the distribution of the S&P 500's distortion is bimodal. In Section 3, we conduct a simulation study to show that such an outcome is not in line with the dynamics of standard linear time series models. In Sections 4, 5 and 6, we present agent-based financial market models in which the distortion possesses a bimodal shape. Section 7 concludes our paper.

2 Distributional properties of the S&P 500's distortion

In this section, we provide visual and statistical evidence demonstrating that the distribution of the S&P 500's distortion is bimodal. Let us start our analysis by inspecting Figure 1. The black line in the top left panel depicts the evolution of the real S&P 500 between January 1871 and December 2015 on a log scale. As can be seen, this price time series, comprising 1,740 monthly observations, is subject to sustained up and down fluctuations around an upward sloping trend. The gray line in this plot represents the S&P 500's fundamental value, as defined by Shiller

(2015). Accordingly, the S&P 500's fundamental value reflects the present value of dividends and is computed from the actual subsequent real dividends using a constant real discount rate of 7.6 percent per year, equal to the historical average real return on the market since 1871. For dividends after December 2015, it is assumed that they will grow forever from the last observed dividend with a growth rate of 5.1 percent per year (which is the dividends' average growth rate between 2004 and 2013).[3] Shiller (2015, p. 249) notes that it is a striking fact that "the present value of dividends looks pretty much like a steady exponential growth line, while the stock market oscillates a great deal around it". Given the evidence, we fully agree with his conclusion.

The top right panel of Figure 1 shows the S&P 500's distortion, i.e. the log difference between the two aforementioned time series. This panel reveals even more clearly how strong the S&P 500's mispricing may be at times. The bottom left panel of Figure 1 depicts a histogram of the S&P 500's distortion, revealing another striking fact. Visual impression suggests that the distribution of the S&P 500's distortion is bimodal, although – as we believe – most economists would expect to see a unimodal distribution. Apparently, this distribution has a local minimum very close to where the market's mispricing is zero, i.e. at the very place where we would expect to find the distribution's global maximum. Our visual impression is further confirmed by the bottom right panel of Figure 1 in which a smoothed histogram of the S&P 500's distortion is presented. While the S&P 500 spends more time in bull markets than in bear markets, the bimodality of its distribution clearly sticks out.[4] Note that the top left panel of Figure 1 already

[3] Shiller's dataset is available at http://www.econ.yale.edu/~shiller/. For more information on the construction of the dataset and on the computation of the S&P 500's fundamental value, see Shiller (1981, 1989, 2015).

[4] As a robustness check, we explored whether the observed bimodality of the distribution of the S&P 500's distortion depends on the level of the real discount rate. Varying the level of the real discount rate between 6.6 and 8.6 percent does not destroy the impression of a bimodal distributed distortion. Note that Boswijk et al. (2007) and Hommes and in't Veld (2016) estimate the agent-based financial market model by Brock and Hommes (1998) using time-varying real discount rates to compute the S&P 500's fundamental value. We stick to Shiller's (1981, 1989, 2015) proposal, but agree with an anonymous referee that more empirical work in this direction would appear to be worthwhile.

suggests that when the S&P 500 turns from a bull market into a bear market or from a bear market into a bull market, it tends to move away from its fundamental value quickly. In Sections 4, 5 and 6, we discuss agent-based models that are able to produce such dynamics.

*** Figure 1 about here ***

We next present statistical evidence that confirms our visual impression. Silverman (1981) devised a test to identify the number of modes of an empirical distribution. The null hypothesis of his test is that the empirical density has at most k modes. Rejecting this hypothesis suggests that the underlying density has more than k modes. The null hypothesis is rejected if the returned p-value is smaller than a given level of significance, say 5 percent. Since the p-value for the null hypothesis that the distribution of the S&P 500's distortion has at most one mode is 0.6 percent and the p-value for the null hypothesis that the distribution of the S&P 500's distortion has at least two modes is 29.0 percent, our visual impression of the bimodality of the distribution of the S&P 500's distortion receives statistical supported.[5] Similar visual and statistical results, albeit slightly less pronounced, are obtained when the dataset is split into two samples of equal size, running from January 1871 to June 1943 and from July 1943 to December 2015. We may thus (carefully) conclude that unimodality can be rejected for the total sample as well as for the two subsamples.[6]

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[5] All tests were carried out using the R package "silvermantest", available at https://www.uni-marburg.de/fb12/stoch/research/rpackage. This package takes Hall and York's (2001) refinements of Silverman's (1981) test into account, preventing it from being too conservative. Note that the test only requires a choice of the number of modes for the null hypothesis (in our case either k=1 or k=2) and the number of bootstrap replications (we used the default setting M=999, but also checked that larger number of repetitions yield similar results). Since critical p-values are derived from varying the bandwidth, the test automatically determines the amount of smoothing.

[6] Since the assumptions behind Silverman's (1981) test are not fully satisfied, our statistical analysis needs to be treated with some care. While our observations are not independent, we remark that the underlying time series is at least stationary. However, this aspect deserves more attention in the future.

3 Distributional properties of standard linear time series models

As is well known, standard linear time series models are not appropriate for explaining the S&P 500's bimodal distributed distortion. Nevertheless, we conduct a simple simulation study to rule out the observed bimodality being due to finite sample effects, assuming that the true distribution is unimodal. First of all, we check which standard linear time series model best explains the dynamics of the S&P 500's distortion.[7] Using the Akaike information criterion as the relevant model selection criterion, we find that an ARMA (2,2) model proves to be the most appropriate linear time series model out of a large class of possible linear models in this respect.[8] A representative simulation run of the estimated ARMA (2,2) model is depicted in the top left panel of Figure 2. This simulation run, containing 1,740 observations, displays fluctuations which at first sight are quite similar to the dynamics of the S&P 500's distortion, as depicted in the top right panel of Figure 1. In particular, the empirical and the simulated time series display strong oscillations around the zero line and spend more time above than below it.

*** Figure 2 about here ***

Nevertheless, there are important differences. The top right panel of Figure 2 shows a smoothed distribution for this simulation run (black line), along with the asymptotic distribution of the estimated ARMA (2,2) model (gray shaded area).[9] As is well known, the distribution of

[7] For our model selection, we use Mathematica's built-in function TimeSeriesModelFit, thereby taking into account AR, MA, ARMA and ARIMA model families. Enders (2014) provides an excellent introduction to this area.

[8] The Bayesian information criterion and the Schwartz-Bayes information criterion both suggest that an ARMA (1,1) model is best at explaining the S&P 500's distortion. Performing the simulation study outlined in this section on the basis of an ARMA (1,1) model reveals almost identical results.

[9] Distributions are smoothed using Mathematica's built-in function SmoothHistogram. The SmoothHistogram function allows choosing a kernel function and a method for the bandwidth selection. We use Mathematica's default setting, i.e. the Gaussian kernel and Silverman's method for the bandwidth selection. However, we also tried other bandwidth selection methods and determined, for instance, the bandwidth in units of standard deviation. Various experiments reveal that our results are robust with respect to the exact smoothing procedure.

ARMA (2,2) models becomes unimodal for sufficiently long samples. For short samples, however, this may not necessarily be the case. For instance, the smoothed distribution of the simulated time series possesses two dips. Compared to the dip we observe in the empirical distribution, however, these dips appear to be much smaller. Since we are able to quantify the magnitude of such dips, we can be more precise. Taking the most conservative approach possible, we measure the magnitude of a dip by taking the smaller of the two distances between the minima and the two maxima. As indicated in the bottom right panel of Figure 1, we obtain a dip size of 0.259 for the empirical distribution. In contrast, the magnitude of the two dips of the simulated distribution amount to only 0.059 and 0.005, respectively.

This brings us to the following test idea. We can use the estimated ARMA (2,2) model to generate a large number of simulation runs. An interesting question is then in how many of these simulation runs we can detect a dip size comparable to the empirical dip. If the fraction of simulation runs in which we notice a dip size exceeding 0.259 is large, the empirical dip does not appear to be particularly surprising since it may be reconciled with the dynamics of standard linear time series models. On the other hand, if we observe such a dip size only rarely, say in less than 5 percent of the simulation runs, we may conclude that the empirical dip cannot be explained by the dynamics of standard linear time series models, and it may therefore be regarded as odd. Of course, in case the simulated distribution has more than one dip, we always compare the empirical dip with the largest dip size of the simulated distribution.

Our simulation study is based on 100,000 simulation runs. One preliminary result is that 26.73 percent of the simulation runs display no dip at all; 49.89 percent of the simulation runs display one dip; 20.19 percent of the simulation runs display two dips; 2.96 percent of the simulations runs display three dips; and 0.22 percent of the simulation runs display four or more dips. More importantly, we obtain the following results with respect to the dip sizes of simulated distributions. The relative frequency distribution of simulated dip sizes is depicted in the bottom left panel of Figure 2. Clearly, the bulk of simulation runs show dip sizes that are much smaller than the empirical dip. The fraction of simulation runs for which the dip size exceeds a critical dip size, i.e. the *p*-value, is presented in the bottom right panel of Figure 2. Accordingly, we observe that the simulated distributions possess a dip size larger than the empirical dip of 0.259 in about p = 1.25 percent of cases.[10] Based on this analysis, we conclude that standard linear time series models are unable to explain the bimodality of the distribution of the S&P 500's distortion. As a byproduct, we note that linear economic dynamics models can be ruled out as potential candidates for explaining this observation. Or, put differently, our simulation study suggests that the S&P 500's distortion may be subject to nonlinear forces.

4 Distributional properties of the agent-based model by Gaunersdorfer and Hommes

We now demonstrate that simple nonlinear agent-based financial market models may account for the bimodality of the S&P 500's distortion. We start our analysis with the model by Gaunersdorfer and Hommes (2007) for the following reasons. First, Gaunersdorfer and Hommes (2007) assume that speculators compute the asset's fundamental value by discounting the sum of future dividends, i.e. the speculators' value approach is closely related to Shiller's value approach, underlying our empirical study. Second, Gaunersdorfer and Hommes (2007) show that their model has the power to explain a number of important stylized facts of financial markets. From this perspective, their model may be regarded as validated. Moreover, we can rely on their parameter setting for our analysis. Third, the dynamics of the deterministic skeleton of their

[10] For the estimated ARMA (1,1) model, mentioned in footnote 8, this probability is about 1.27 percent. In contrast, resampling the S&P 500's distortion 100,000 times via a block bootstrap with block lengths of 10 years reveals that dips exceeding the empirical dip can be detected in about 50 percent of these time series.

calibrated model gives rise to endogenous dynamics, which reveals most clearly why the distribution of a stock market's distortion may be bimodal. After exploring the model by Gaunersdorfer and Hommes (2007), we discuss – as a robustness check – the model by Franke and Westerhoff (2012) in Section 5 and the models by Zeeman (1974), Day and Huang (1990), Chiarella (1992), de Grauwe et al. (1993), Lux (1995) and Brock and Hommes (1998) in Section 6. Note that we explicitly abstain from building our own agent-based financial market model. While it may be possible to develop a model that fits the dynamics of stock markets even better, we find it more insightful and convincing to show that the puzzling behavior of the S&P 500's distortion is already solved by existing agent-based models.

Let us turn to the details of the model by Gaunersdorfer and Hommes (2007). Their model represents a standard discounted value asset pricing model, with the exception that speculators rely on technical and fundamental analysis rules to predict future asset prices. Speculators' predictor selection depends on the rules' past profitability and on current market conditions. Speculators can invest in a risky asset, paying an uncertain dividend y_t per share, and in a risk-free asset, paying a fixed rate of return r. Gaunersdorfer and Hommes (2007) assume an IID dividend process for the risky asset, specified as $y_{t+1} = \bar{y} + \delta_{t+1}$ with $\delta_{t+1} \sim N(0, \sigma^{\delta})$. Speculators are aware of the properties of the dividend process and compute the risky asset's fundamental value by discounting the sum of expected future dividends. Accordingly, the fundamental value of the risky asset is perceived as $p^* = \bar{y} / r$. Note that if all speculators had rational expectations, the model's rational expectation equilibrium price would be p^*.

Speculators using technical analysis are called chartists and are indexed by $h = C$; speculators using fundamental analysis are called fundamentalists and are indexed by $h = F$. Forecasts according to the technical analysis rule are expressed as

$$E_t^C[p_{t+1}] = p_{t-1} + g(p_{t-1} - p_{t-2}), \tag{1}$$

where $g \geq 0$ denotes the strength of speculators' trend extrapolation. Predictions of the fundamental analysis rule are formalized as

$$E_t^F[p_{t+1}] = p^* + v(p_{t-1} - p^*), \tag{2}$$

where $0 \leq v \leq 1$ indicates speculators' expected mean reversion speed. The time structure of this model implies that speculators do not know the current price of the risky asset when they form their predictions. The last observable information that enters (1) and (2) is from period $t-1$.

Speculators are myopic mean-variance maximizers. As is well known, the demand for the risky asset by speculator type h is thus given by

$$z_t^h = \frac{E_t^h[p_{t+1} + y_{t+1} - (1+r)p_t]}{a\, V_t^h[p_{t+1} + y_{t+1} - (1+r)p_t]}, \tag{3}$$

where a denotes the speculators' uniform risk aversion parameter. For simplicity, Gaunersdorfer and Hommes (2007) assume that the beliefs about the conditional variance in (3) are constant over time and equal for all speculators, i.e. $V_t^h[p_{t+1} + y_{t+1} - (1+r)p_t] = \sigma^2$.[11]

The price of the risky asset adjusts such that the market clears in every period. Let z^s denote the outside supply of the risky asset per speculator and n_t^C and n_t^F the market shares of chartists and fundamentalists, respectively. The market equilibrium condition can then be expressed as $n_t^C E_t^C[p_{t+1} + y_{t+1} - (1+r)p_t]/a\sigma^2 + n_t^F E_t^F[p_{t+1} + y_{t+1} - (1+r)p_t]/a\sigma^2 = z^s$. Assuming that the outside supply of the risky asset is zero yields the price equation

[11] By exploring the more complicated case in which speculators' variance beliefs are time-varying, Gaunersdorfer (2000) shows that this simplifying assumption is, at least in the case of an IID dividend process, not critical for the model's key results.

$$p_t = \frac{1}{1+r}(n_t^C E_t^C[p_{t+1}] + n_t^F E_t^F[p_{t+1}] + \bar{y}) + \varepsilon_t, \tag{4}$$

where the random variable $\varepsilon_t \sim N(0, \sigma^\varepsilon)$ captures additional effects on p_t that are not explicitly considered in the model, such as exogenous liquidity demand or the market impact of noise traders.[12]

Accordingly, the risky asset's price equals the discounted value of tomorrow's average expected price plus tomorrow's expected dividends. It remains to specify how the market shares of chartists and fundamentalists change over time. Note first that accumulated realized profits by speculator type h result in

$$U_t^h = (p_t + y_t - (1+r)p_{t-1})z_{t-1}^h + \eta U_{t-1}^h. \tag{5}$$

The first term on the right-hand side of (5) represents current realized excess profits of speculator type h which are given by the realized excess return per share of the risky asset over the risk-free asset multiplied by the demand for the risky asset. The second term on the right-hand side of (5) denotes speculator type h's past accumulated realized profits, with $0 \le \eta \le 1$ being a memory parameter that measures how quickly past performance is discounted.[13]

Gaunersdorfer and Hommes (2007) use the discrete choice approach by Manski and McFadden (1981) to model the market shares of chartists and fundamentalists, yet buffet it by a correction term. To be precise, the market share of chartists is given by

$$n_t^C = \frac{\exp[\beta U_{t-1}^C]}{\exp[\beta U_{t-1}^C] + \exp[\beta U_{t-1}^F]} \cdot \exp[-\frac{(p^* - p_{t-1})^2}{\alpha}]. \tag{6}$$

Since the market shares of chartists and fundamentalists add up to one, the market share of

[12] Note that Brock (1997) motivates the simplifying case $z^s = 0$ by introducing risk-adjusted dividends.

[13] Gaunersdorfer et al. (2008) provide an analysis of this model with risk-adjusted profits. Their results are similar to those generated by Gaunersdorfer and Hommes (2007).

fundamentalists can be expressed as

$$n_t^F = 1 - n_t^C \,. \tag{7}$$

The discrete choice term in (6) ensures that the rule selection behavior of speculators depends on the rules' profitability, where parameter $\beta \geq 0$ controls how quickly the mass of speculators switches to the more profitable rule. Due to the correction term in (6), speculators' rule selection behavior also depends on current market conditions. Parameter $\alpha > 0$ regulates how sensitively the mass of speculators reacts to the market's mispricing. As long as the price of the risky asset is close to its fundamental value, the market fractions of the speculators' rules depend almost completely on their past profitability. But if the price of the risky asset moves away from its fundamental value, the correction term becomes smaller. More and more speculators then start to believe that a fundamental price correction is about to occur.

The dynamics of the model by Gaunersdorfer and Hommes (2007) is due to a six-dimensional first-order nonlinear dynamical system. Gaunersdorfer and Hommes (2007) show that their model can explain a number of stylized facts of financial markets, including bubbles and crashes, excess volatility, fat-tailed return distribution, uncorrelated price changes and volatility clustering.[14] For reviews of the statistical properties of financial markets see, among others, Mantegna and Stanley (2000), Cont (2001) and Lux and Ausloos (2002). The top left panel of Figure 3 shows a representative price trajectory of the model. Recall that Gaunersdorfer

[14] Bearing in mind the model's ability to match the stylized facts of financial markets, Franke (2009) extends this model along two lines. First, he considers multiplicative noise instead of additive noise in price equation (4). Second, he adds random shocks to expectation rules (1) and (2). Straightforward simulations reveal that these two model versions also give rise to bimodal distributed distortions. Since the original model by Gaunersdorfer and Hommes (2007) allows a better explanation of the origins of the distortion's bimodality, we base the bulk of our analysis on it. However, the model by Franke and Westerhoff (2012), discussed in Section 5, incorporates Franke's (2009) suggestions.

and Hommes (2007) calibrate their model to the daily behavior of the S&P 500. To obtain 1,740 monthly observations, as in our empirical analysis, we simulate their model for a time span of 145 years, assuming that a year gives rise to $21 \cdot 12 = 252$ trading days. Out of these $145 \cdot 252 = 36{,}540$ daily observations, we plot every 21st observation. Of course, the simulation run is based on the parameter setting by Gaunersdorfer and Hommes (2007), that is we set $\bar{y} = 1$, $\sigma^{\delta} = 0$, $r = 0.001$, $v = 1$, $g = 1.9$, $a\sigma = 1$, $\sigma^{\varepsilon} = 10$, $\eta = 0.99$, $\alpha = 1{,}800$ and $\beta = 2$. Note that the price of the risky asset (black line) oscillates in wild swings around its fundamental value (gray line), similar to the case of the S&P 500.

*** Figure 3 about here ***

The top right panel of Figure 3 depicts the distribution of the simulation run's distortion, again defined as the log difference between the risky asset's price and its fundamental value, while the bottom left panel in Figure 3 shows the asymptotic distribution of the model's distortion, derived from a sufficiently long time series. Evidence for bimodality is clearly visible. In absolute terms, the risky asset's price dynamics is (almost) symmetric with respect to its fundamental value. In relative terms, however, bear markets may be more severe than bull markets. Overall, we find it surprising how well the model by Gaunersdorfer and Hommes (2007) matches the magnitude of bubbles and crashes and, in particular, the magnitude of the distortion – although it was not designed for this specific purpose. The bottom right panel depicts the evolution of the price of the risky asset versus the market share of chartists. We return to this panel in the sequel.

Figure 4 illustrates the deterministic dynamics of the calibrated model by Gaunersdorfer and Hommes (2007), i.e. for $\sigma^{\delta} = 0$ and $\sigma^{\varepsilon} = 0$.[15] The top left panel of Figure 4 reveals that the

[15] Gaunersdorfer and Hommes (2007) show that the deterministic skeleton of their model has a unique steady state in

deterministic model gives rise to a limit cycle according to which the price of the risky asset (black line) oscillates around its fundamental value (gray line). The top right panel of Figure 4 shows the corresponding market shares of chartists. The dynamics may be understood as follows. Around period 100, the price of the risky asset is considerably below its fundamental value. As a result, the market impact of chartists is low. Due to the trading behavior of fundamentalists, the price of the risky asset recovers slowly.[16] This has two important effects. First, technical analysis correctly predicts the upward movement of the market and thus generates profits. Second, the market's mispricing decreases continuously. Both effects increase the market share of chartists which, in turn, amplifies the momentum of the market's upward movement. Eventually, however, the price of the risky asset overshoots its fundamental value, and speculators switch to fundamental analysis. Now the dynamics of the model reverses. Initially, the price of the risky asset decreases slowly. Then the momentum of the price reduction picks up, until the market finally crashes. Overall, this kind of dynamics – a rapid movement away from the fundamental value combined with a slow correction of high misalignment levels – renders the distribution of the market's distortion bimodal, as evidenced by the bottom left panel of Figure 4.

*** Figure 4 about here ***

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which the price of the risky asset is equal to its fundamental value and in which the market shares of chartists and fundamentalists are equal. Moreover, they show that the steady state becomes unstable if chartists' trend chasing becomes sufficiently strong, i.e. if $g > 2\ (1 + r)$. The dynamics is then characterized by periodic, quasi-periodic or chaotic motion. The model may also give rise to endogenous dynamics for $g < 2\ (1 + r)$. For the current parameter setting, for instance, the locally stable steady state coexists with a locally stable limit cycle.

[16] The calibration of the model suggests that v is close to unity. In fact, Gaunersdorfer and Hommes (2007) set v equal to 1, which implies that the fundamental analysis rule predicts no change in the risky asset's price. Fundamentalists may thus be regarded as efficient market believers. Note that if all speculators rely on fundamental analysis, the dynamics of the deterministic model is due to $p_t = p_{t-1} + (r/(1+r))\ (p^* - p_{t-1}) \approx p_{t-1} + 0.001\ (p^* - p_{t-1})$, indicating that the price converges only very slowly to its fundamental value.

The bottom right panel of Figure 4 depicts the development of the price of the risky asset versus the market share of chartists. The dynamics on this figure-8 cycle is counterclockwise. When the market is slightly overvalued and the market impact of chartists is high, the price of the risky asset disconnects rapidly from its fundamental value. When the market is strongly overvalued and the market impact of chartists is low, the price of the risky asset needs a considerable amount of time to converge towards its fundamental value. However, the price dynamics becomes faster and faster and finally overshoots the fundamental value. Note that such a counterclockwise figure-8 dynamics is also discernible in the bottom right panel of Figure 3. While exogenous shocks make the attractor fuzzier, the same forces are at work as in the deterministic model. One important difference is that exogenous shocks amplify the fluctuations of the price dynamics. In some periods, the market may be strongly overvalued or undervalued and then it may take even longer for the market to normalize. Obviously, this reinforces the bimodality of the distribution of the market's distortion.

5 Distributional properties of the agent-based model by Franke and Westerhoff

The model by Franke and Westerhoff (2012) differs from the model by Gaunersdorfer and Hommes (2007) along a number of important dimensions. First, Franke and Westerhoff (2012) assume a disequilibrium framework in which a market-maker changes the price of the risky asset with respect to speculators' order flow. Second, there are neither direct price shocks nor dividend shocks. Instead, Franke and Westerhoff (2012) assume that speculators' trading rules entail random elements. Third, speculators' rule selection behavior depends on predisposition effects, herding behavior and market misalignments. Put differently, past profits do not influence speculators' rule selection behavior. Finally, Franke and Westerhoff (2012) estimate their model using the method of simulated moments. Their model is supported by the data, and matches the statistical properties of the daily behavior of the S&P 500 quite well, also from a quantitative perspective.

Let us briefly summarize the main building blocks of this model (as in the previous case, we follow the authors' notation to present the model). Franke and Westerhoff (2012) assume that the market-maker quotes the next period's price of the risky asset with respect to the speculators' excess demand, using the log-linear price adjustment rule

$$p_{t+1} = p_t + \mu(n_t^C d_t^C + n_t^F d_t^F), \tag{8}$$

where p_t is the log of the price of the risky asset, $\mu > 0$ is the market-maker's price adjustment parameter, n_t^C and n_t^F are the market shares of chartists and fundamentalists (the number of speculators is normalized to one), and d_t^C and d_t^F are the orders placed by a single chartist and a single fundamentalist, respectively. Accordingly, the market-maker increases (decreases) the log of the price of the risky asset if the speculators' excess demand is positive (negative).

Franke and Westerhoff (2012) assume that speculators have the choice between a representative technical trading rule and a representative fundamental trading rule. Orders generated by these two trading rules are formalized as

$$d_t^C = \chi(p_t - p_t) + \varepsilon_t^C \tag{9}$$

and

$$d_t^F = \phi(p^* - p_t) + \varepsilon_t^F, \tag{10}$$

where χ and ϕ are positive reaction parameters, p^* is the log of the fundamental value, and $\varepsilon_t^C \sim N(0, \sigma^C)$ and $\varepsilon_t^F \sim N(0, \sigma^F)$ are random disturbance terms. Note that both (9) and (10) entail a deterministic and a stochastic component. The deterministic components reflect the basic principle of technical and fundamental analysis. While technical analysis predicts that the price of the risky asset moves in trends, fundamental analysis presumes that the price of the risky asset returns towards its fundamental value. The stochastic components capture part of the diversity of actual technical and fundamental trading rules.[17]

The market share of speculators following the technical trading rule is given by

$$n_t^C = \frac{\exp[\beta u_{t-1}^C]}{\exp[\beta u_{t-1}^C] + \exp[\beta u_{t-1}^F]} = \frac{1}{1+\exp[\beta(u_{t-1}^F - u_{t-1}^C)]} = \frac{1}{1+\exp[\beta a_{t-1}]} . \tag{11}$$

Since the market shares of chartists and fundamentalists add up to one, we further have

$$n_t^F = 1 - n_t^C . \tag{12}$$

Obviously, the market shares of chartists and fundamentalists are modeled via the discrete choice approach, where β stands for the speculators' intensity of choice and u_{t-1}^C and u_{t-1}^F denote the fitness of the trading rules. Note that the discrete choice approach implies that the rules' relative fitness, defined as $a_{t-1} = u_{t-1}^F - u_{t-1}^C$, is what matters for speculators' rule selection behavior.

Franke and Westerhoff (2012) model the relative fitness of fundamental analysis over technical analysis as

$$a_t = \alpha_0 + \alpha_n (n_t^F - n_t^F) + \alpha_p (a_t - p^*)^2 . \tag{13}$$

Accordingly, the relative fitness depends on three socio-economic principles. First, speculators may display a behavioral preference for one of the two trading rules, expressed by parameter α_0. If $\alpha_0 > 0$, speculators have a behavioral preference for fundamental analysis. If $\alpha_0 < 0$, they have a behavioral preference for technical analysis. Second, speculators are subject to herding behavior. The higher the herding parameter $\alpha_n > 0$, the more strongly speculators follow the

[17] Schmitt and Westerhoff (2016) show that the representative trading rules (9) and (10) may, under some assumptions, be derived from a setup in which all speculators follow their individual technical and fundamental trading rules.

crowd. Third, speculators react to current market conditions. As the price of the risky asset moves away from its fundamental value, they perceive an increasing probability that a fundamental price correction is about to set in, thus preferring more strongly fundamental analysis over technical analysis. Parameter $\alpha_p > 0$ controls the extent to which the relative fitness depends on current market conditions.

Since the market-maker's price adjustment parameter and the speculators' intensity of choice parameter are scaling parameters, they are set to $\mu = 0.01$ and $\beta = 1$. Moreover, the dynamics of the model does not depend on the level of the log of the fundamental value and thus it is assumed that $p^* = 0$. Franke and Westerhoff (2012) estimate the remaining seven model parameters using the method of simulated moments. The method of simulated moments searches for the parameter setting for which the model best matches a predefined set of summary statistics, capturing important stylized facts of financial markets. The estimated parameter setting, given by $\chi = 1.5$, $\phi = 0.12$, $\sigma^C = 2.147$, $\sigma^F = 0.708$, $\alpha_0 = -0.336$, $\alpha_n = 1.839$ and $\alpha_p = 19.671$, reveals that speculators have a behavioral preference for technical analysis and that the randomness associated with the technical trading rules is about three times as strong as the randomness associated with the fundamental trading rule. In addition, $\alpha_p = 19.671$ signifies that speculators' rule selection behavior depends on market circumstances.

The summary statistics included in the estimation approach by Franke and Westerhoff (2012) explicitly consider the S&P 500's average volatility, its fat-tailed return distribution, its random walk property and its volatility clustering behavior. Although the estimation approach does not account for the S&P 500's distortion, the top left panel of Figure 5 reveals that the model is able to produce bubbles and crashes.[18] As can be seen, the log of the price of the risky asset (black line) fluctuates in a complex manner around its log fundamental value (gray line).[19] The top right panel of Figure 5 shows the distribution of the distortion for this simulation run, while the bottom left panel of Figure 5 shows the asymptotic distribution of the model's distortion (computed from a sufficiently long time series). Although the magnitude of the model's boom and bust dynamics is less pronounced than the one of the S&P 500, the distribution of the model's distortions has – without question – a clear bimodal shape.

*** Figure 5 about here ***

The explanation for this is as follows. Suppose that the price of the risky asset is close to its fundamental value. In such a situation, the market is dominated by chartists and their trading behavior rapidly drives the price of the risky asset away from its fundamental value. But the more severe the market's distortion becomes, the more speculators prefer fundamental analysis. Since the mean reversion pressure exercised by fundamentalists is rather weak, the price of the risky asset converges only slowly towards its fundamental value. Once the market's distortion has resolved, speculators return towards destabilizing technical trading rules and create the next boom or bust episode. This can also be seen in the bottom right panel of Figure 5 in which the market's distortion is plotted against the market share of chartists. For visibility reasons, we project only a snapshot of the dynamics in this panel. Broadly speaking, there are two regimes. If the market's distortion is low, the market impact of chartists is strong and the price is quickly

[18] Franke and Westerhoff (2012) calibrate their model to the daily behavior of the S&P 500. We thus simulate $21 \cdot 12 \cdot 145 = 36{,}540$ daily observations of which every 21st observation is used to obtain 1,740 monthly observations.

[19] The estimated parameter setting implies that the deterministic version of this model possesses a locally stable steady state in which prices reflect their fundamental values. The (technical) reason why the stochastic version of this model produces realistic dynamics is that exogenous noise triggers complex transient dynamics. Franke and Westerhoff (2016) provide a deeper analysis of the interplay between a locally stable steady state, exogenous noise, complex transient dynamics and the stylized facts of financial markets for a closely related model.

driven away from fundamental values. Alternatively, if the market's distortion is high, the market impact of fundamentalists is strong and the price slowly approaches its fundamental value. Since the latter regime is more persistent, the dynamics spends relatively more time in bull or bear markets than in the vicinity of its fundamental value.[20] Note that the strong destabilizing nature of technical analysis, the weak stabilizing nature of fundamental analysis, and the speculators' preference for fundamental analysis in distorted markets are not convenient ad hoc assumptions – the model by Franke and Westerhoff (2012) has been estimated and thus these model features are empirically supported.

6 Distributional properties of other agent-based financial market models

In the last two sections, we discussed two empirically validated agent-based financial market models that are able to explain the bimodality of the distribution of the S&P 500's distortion. The goal of this section is twofold. First, we show that a number of seminal agent-based financial market models, including the contributions by Zeeman (1974), Day and Huang (1999), Chiarella (1992), de Grauwe et al. (1993), Lux (1995) and Brock and Hommes (1998), are also able, at least from a qualitative perspective, to produce a bimodal distributed distortion. In contrast to the

[20] In the models by Gaunersdorfer and Hommes (2007) and Franke and Westerhoff (2012), the fundamental value plays a prominent role: it determines the demand of fundamentalists and influences speculators' rule selection behavior. As a robustness check, we also considered the case in which speculators misperceive the fundamental value by assuming that $F_t = \bar{F} + \varepsilon_t$, where $\bar{F}$ is the true fundamental value and $\varepsilon_t \sim N(0, \sigma)$ captures speculators' perception errors. The models by Gaunersdorfer and Hommes (2007) and Franke and Westerhoff (2012) still produce bimodal distributed distortions, even if speculators' perception errors become large. More elaborate models for speculators' perception of the fundamental value, such as AR(1) models, yield similar results. The reason for this – at least at first sight – surprising result is that the stock market's mispricing is usually quite substantial in both models, and it does not matter much whether a market is, say, 30 or 35 percent overvalued. Since Shiller (2015) demonstrates that the S&P 500 is also quite mispriced on average, we conjecture that fundamental perception errors, or disagreement about the true fundamental value, are also not that relevant for the dynamics of actual stock markets. For the models discussed in Section 6, however, the fundamental value is less relevant.

models by Gaunersdorfer and Hommes (2007) and Franke and Westerhoff (2012), these models depend less strongly on the assumption that speculators know the stock market's true fundamental value. Second, we show that quite different dynamical mechanisms may give rise to a bimodal distributed distortion. In the model by Franke and Westerhoff (2012), the underlying unique steady state is locally stable. The distribution of the distortion is bimodal since exogenous shocks trigger complex transient dynamics. There are two locally stable steady states in the model by Zeeman (1974), as well as in the models by Day and Huang (1990), Lux (1995) and Brock and Hommes (1998). A bimodal distributed distortion may emerge in these models if exogenous noise induces random transitions between the two attracting steady states. In the models by Chiarella (1992), Lux (1998) and Gaunersdorfer and Hommes (2007), the bimodal distributed distortion is due to a limit cycle. In Brock and Hommes (1998), two coexisting locally stable limit cycles, subjected to exogenous shocks, can yield random changes between bull and bear market dynamics. Finally, the models by Day and Huang (1990) and de Grauwe et al. (1993) generate chaotic price fluctuations, which, in turn, lead to a bimodal distributed distortion.

In the following, we briefly recap the essence of these models in chronological order. The cusp catastrophe model by Zeeman (1974) is one of the first contributions that takes into account the behavior of chartists and fundamentalists. One property of Zeeman's (1974) model is that it may possess three coexisting steady states: an unstable inner steady state and two locally stable outer steady states. Moreover, the inner steady state marks the border of the basins of attraction of the two outer steady states. Another important property of this model is that the adjustment towards the outer steady states is assumed to occur very fast. While Zeeman (1974) models the change of a stock market index, we follow Diks and Wang (2016) and use this model to explain the level of a stock market index. To be precise, the log of the stock market index results as

$$P_{t+1} = \begin{cases} P^H + \varepsilon_t & if \quad P_t > F \\ F + \varepsilon_t & if \quad P_t = F \\ P^L + \varepsilon_t & if \quad P_t < F \end{cases}$$, where F is the log of the fundamental value of the stock market, corresponding to the unstable inner steady state, P^H and P^L are locally stable bull and bear market steady states with $P^H > F > P^L$, and ε_t captures a normally distributed random disturbance term with mean zero and constant standard deviation σ. Due to the instantaneous adjustment towards the bull or bear market steady state, the log of the stock market index is (basically) either given by $P^H + \varepsilon_t$ or by $P^L + \varepsilon_t$, depending on whether the stock market is above or below the fundamental value. The top left panel of Figure 6 shows the evolution of the log of the stock market index for 300 periods. As indicated by the top right panel of Figure 6, random switches between bull and bear market dynamics imply a bimodal distributed distortion. The underlying parameter setting for these simulations is given by $P^H = 0.25$, $F = 0$, $P^L = -0.25$ and $\sigma = 0.1$. Also Diks and Wang (2016), who study the dynamics of housing markets, remark that the model by Zeeman (1974) is able to produce a bimodal distributed distortion. Rosser (2007) provides more historical background on Zeeman's (1974) model while Barunik and Vosvrda (2009) present empirical support for Zeeman's (1974) model.

*** Figure 6 about here ***

The model by Day and Huang (1990) follows the tradition of Zeeman (1974), but is more explicit when it comes to describing the market participants' trading behavior. Day and Huang (1990) assume that a market-maker adjusts the stock price using the linear price adjustment function $P_{t+1} = P_t + a(D_t^C + D_t^F)$, where a stands for a positive price adjustment parameter and D_t^C and D_t^F denote the orders placed by chartists and fundamentalists, respectively. Chartists believe in the persistence of bull and bear markets, i.e. they buy (sell) when the stock price is above (below) its fundamental value. Their orders are represented by the linear trading rule $D_t^C = b(P_t - F)$, where b is a positive reaction parameter. Note that the price dynamics in this model is bounded between 0 and 1 and that the fundamental value is assumed to be $F = 0.5$. Fundamentalists believe in mean reversion and use the trading rule $D_t^F = \frac{c(F-P_t)}{\sqrt{(P_t+0.01)(0.99-P_t)}}$. The main motivation behind the nonlinearity of this trading rule is as follows. Fundamentalists become more and more convinced that a fundamental price correction is about to set in as the market's misalignment increases. Therefore, they trade more aggressively as the price deviates from its fundamental value. The bottom left panel of Figure 6 shows a simulation run with 80 observations for this model, assuming that $a = 1$, $b = 0.89$, c=0.2 and $F = 0.5$. Apparently, the model may endogenously produce alternating periods of bull and bear market dynamics. As evidenced by the bottom right panel of Figure 6, such chaotic dynamics may give rise to a bimodal distributed distortion.[21] Since the dynamics is bounded between 0 and 1, the distortion in in this panel is computed as the difference between the stock price and its fundamental value. For recent extensions of the model by Day and Huang (1990), see Huang et al. (2010), Huang and Zheng (2012) and Tramontana et al. (2013). Amongst others, these contributions make it clear that the model by Day and Huang (1990) has the potential to match a number of important stylized facts of financial markets.

In contrast to Day and Huang (1990), Chiarella (1992) assumes that the trading behavior of fundamentalists is linear while the trading behavior of chartists is nonlinear. There are several motivations for the chartists' nonlinear trading behavior. For instance, Chiarella et al. (2011)

[21] The model by Day and Huang (1990) may also yield locally stable bull and bear market steady states. In these steady states, destabilizing orders placed by chartists are just offset by stabilizing orders placed by fundamentalists. Buffeted with dynamic noise, the dynamics may, as in Zeeman's (1974) model, produce a bimodal distributed distortion. Two further comments are in order. Due to their close relation, the model by Day and Huang (1990) may be used to better understand the origin of the multiple steady states in Zeeman's (1974) model. Moreover, the discrete-time approximation of Zeeman's (1974) model presented in Diks and Wang (2016) may generate endogenous bull and bear market dynamics which are similar to the bull and bear market dynamics reported in Day and Huang (1990).

argue that chartists become more cautious in their trend-extrapolating behavior as price trends accelerate. Moreover, they use a hyperbolic tangent function to model the generally sigmoid demand function of chartists. Although originally formulated in continuous time, we express the model in discrete time. Using a log-linear price adjustment rule, the log of the price is determined by $P_{t+1} = P_t + a(D_t^C + D_t^F)$, where a is a positive price adjustment parameter and D_t^C and D_t^F are the excess demands of chartists and fundamentalists, respectively. In its most simple form, the excess demand of chartists can be expressed as $D_t^C = b \tanh(c(P_t - P_{t-1}))$ while the excess demand of fundamentalists takes the form $D_t^F = d(F - P_t)$, where b, c and d are positive parameters and F denotes the log fundamental value. The top left panel of Figure 7 displays a simulation run for the parameter setting $a = 1$, $b = 1$, $c = 1.05$, $d = 0.85$ and $F = 0$. The log price dynamics is depicted for 60 periods and reveals that the model produces a limit cycle. The top right panel of Figure 7 shows the corresponding distribution of the distortion. Note that Chiarella et al. (2008, 2011) already point out that this framework may give rise to a bimodal distributed distortion. In particular, they are interested in the stochastic bifurcation behavior of a noisy version of this model and study under which conditions so-called phenomenological and dynamical bifurcations may turn a unimodal distribution into a bimodal distribution. For more technical details about these kinds of bifurcation and their distributional implications, see Diks and Wagener (2008, 2011).

*** Figure 7 about here ***

In spite of having been developed to explain the dynamics of foreign exchange markets, the model by de Grauwe et al. (1993) may also be applied to stock markets. In the basic version of the model, the current stock price is given by $P_t = X_t(E_t[P_{t+1}])^a$, where X_t is an exogenous variable representing the fundamental part of the model, $E_t[P_{t+1}]$ is the average of the speculators' next period's expected stock price, and parameter $0 < a < 1$ is a discount factor. We follow de Grauwe et al. (1993) and set $X_t = 1$, implying that the model's fundamental value results in $F = 1$. Market participants either form regressive expectations, i.e. $E_t^F[P_{t+1}] = P_{t-1}(F/P_{t-1})^b$, where $0 < b < 1$ indicates the fundamentalists' mean reversion speed, or extrapolative expectations, i.e. $E_t^C[P_{t+1}] = P_{t-1}((\frac{P_{t-1}}{P_{t-2}})/(\frac{P_{t-1}}{P_{t-3}})^{0.5})^{2c}$, where $c > 0$ captures the strength of chartists' extrapolation behavior. The latter rule combines a short-term moving average, given by (P_{t-1}/P_{t-2}), with a long-term moving average, given by $(P_{t-1}/P_{t-3})^{0.5}$, and predicts, for instance, a price increase (decrease) if the short-term moving average is more (less) positive than the long-term moving average. An interesting feature of the model by de Grauwe et al. (1993) is that fundamentalists disagree about the stock market's true fundamental value. To be precise, the fundamentalists' perception of the fundamental value is normally distributed around the true fundamental value. As a result, half of the fundamentalists believe that the stock market is overvalued (undervalued) when the price equals its fundamental value. In such a situation, the market impact of fundamentalists is zero. However, the more the price exceeds (falls below) the fundamental value, the more fundamentalists are convinced that the stock market is overvalued (undervalued). Hence, the fundamentalists' market impact increases with the stock market's mispricing. For this reason, de Grauwe et al. (1993) define the speculators' average stock price expectation as $E_t[P_{t+1}] = E_t^C[P_{t+1}]^{m_t} E_t^F[P_{t+1}]^{1-m_t}$, where the market impact of chartists and fundamentalists is due to the bell-shaped function $m_t = 1/(1 + d(F - P_{t-1})^2)$. Parameter $d > 0$ controls how quickly the market impact of fundamentalists increases as the stock price moves away from its fundamental value. The bottom left panel of Figure 7 depicts a time series example for this model with 300 observations, assuming that $a = 0.95, b = 0.65,\ c = 3,\ d = 10{,}000$ and $F = 1$. The dynamics is characterized by chaotic stock price fluctuations. As revealed by the bottom right panel of Figure 7, the associated distribution of the distortion is bimodal.

Lux (1995) stresses the relevance of socio-economic interactions and sentiment dynamics. In his model, chartists are either optimistic or pessimistic. Most importantly, the mood of chartists depends on herding behavior. For instance, the probability that a pessimistic chartist will become an optimistic chartist increases with the number of optimistic chartists. Furthermore, the mood of chartists depends on market conditions. For instance, the probability that a pessimistic chartist will become an optimistic chartist increases if stock prices increase. For simplicity, we present the model by Lux (1995) in discrete time, assume that there is a continuum of chartists, normalized to $N = 1$, and allow for some exogenous noise. Accordingly, a market-maker quotes the stock price using the price adjustment rule $P_{t+1} = P_t + a(N_t^O D_t^O + N_t^P D_t^P + D_t^F) + \varepsilon_t$, where a is a positive price adjustment parameter, D_t^O and D_t^P are the orders placed by optimistic and pessimistic chartists, N_t^O and N_t^P are the market shares of optimistic and pessimistic chartists (with $N_t^P = 1 - N_t^O$), D_t^F are the orders placed by fundamentalists, and ε_t is normally distributed exogenous noise with mean zero and constant standard deviation σ. The orders placed by optimistic and pessimistic chartists are formalized as $D_t^O = b$ and $D_t^P = -b$ with $b > 0$, i.e. they buy or sell fixed amounts of stocks. The orders placed by fundamentalists are represented by $D_t^F = \mathrm{c}(F - P_t)$, where c is a positive reaction parameter and F is the fundamental value. The market share of optimistic chartists evolves as $N_t^O = N_{t-1}^O + N_{t-1}^P \pi_t^{PO} + N_{t-1}^O \pi_t^{OP}$, where $\pi_t^{PO} = \min[d \exp[A_t], 1]$ is the probability that a pessimistic chartist will turn optimistic and $\pi_t^{OP} = \min[d \exp[-A_t], 1]$ is the probability that an optimistic chartist will turn pessimistic. Parameter d is positive and $A_t = e\left(N_{t-1}^O - N_{t-1}^P\right) + f(P_{t-1} - P_{t-2})$ may be regarded as the relative fitness of an optimistic chartist mood over a pessimistic chartist mood. Since $e > 0$, the probability that a pessimistic chartist will become optimistic increases with the number of optimistic chartists. Due to $f > 0$, the probability that a pessimistic chartist will become optimistic also increases if prices increase more strongly. As long as the mean reversion trading of fundamentalists is not too pronounced, an increase in the number of optimistic chartists will drive the stock market up.

The model by Lux (1995) can produce very rich dynamics; we discuss two different parameter constellations. The top left panel of Figure 8 shows a simulation run for 1,000 periods, assuming that $a = 1$, $b = 1.7$, $c = 0.6$, $d = 0.5$, $e = 1.1$, $f = 0.4$, $\sigma = 0.2$ and $F = 5$. This parameter setting gives rise to two coexisting locally stable steady states, one above and one below the fundamental value. Due to exogenous noise, however, the model produces erratic switches between bull and bear market regimes. The corresponding distribution of the distortion, depicted in the top right panel of Figure 8, is bimodal. The bottom left panel of Figure 8 presents a simulation run for 200 periods, assuming that $a = 1$, $b = 1.8$, $c = 0.5$, $d = 0.5$, $e = 1.2$, $f = 1.2$, $\sigma = 0.1$ and $F = 5$. In the absence of exogenous noise, the model now generates a stable limit cycle. Adding some exogenous noise, however, renders the dynamics somewhat more irregular. As can be seen in the bottom right panel of Figure 8, the distribution of the distortion is again bimodal. The model by Lux (1995) is related to the herding model by Kirman (1991, 1993). In his model, agents have the choice between two options and agents' choices are subject to their social interactions. Without going into too much detail, Kirman (1991, 1993) shows that the distribution of agents across the two options may be bimodal. Assuming that one option implies taking a long position while the other option implies taking a short position, it is easy to envision a further financial market environment that gives rise to a bimodal distributed distortion.

*** Figure 8 about here ***

Brock and Hommes (1998) propose an agent-based financial market model in which speculators switch between different prediction rules to forecast asset prices. In particular, the speculators' rule selection behavior depends on the past performance of the rules.[22] The model by

[22] The model by Gaunersdorfer and Hommes (2007) is closely related to the model by Brock and Hommes (1998). We therefore keep the presentation of the model by Brock and Hommes (1998) rather short.

Brock and Hommes (1998) can be formulated in deviations from fundamental values. Accordingly, the price of the risky asset (in deviations from its fundamental value) is equal to the discounted value of the speculators' average price expectation for the next period plus an exogenous noise term, i.e. $X_t = (1/(1+r))(N_t^C E_t^C[X_{t+1}] + N_t^F E_t^F[X_{t+1}]) + \varepsilon_t$, where r is the risk-free interest rate, N_t^C and N_t^F are the market shares of speculators using extrapolative and regressive expectations, and ε_t is a normally distributed random variable with mean zero and constant standard deviation σ. Extrapolative expectations are given by $E_t^C[X_{t+1}] = aX_t$ with $a > 1$ and predict, as in the model by Day and Huang (1990), an increase in the deviation between prices and fundamental values. Regressive expectations are formalized by $E_t^F[X_{t+1}] = 0$ and forecast a prompt return of prices towards fundamental values. The market shares of chartists and fundamentalists are determined by the discrete choice approach, that is $N_t^C = \exp[bU_{t-1}^C]/(\exp\left[bU_{t-1}^C\right] + \exp[bU_{t-1}^F])$ and $N_t^F = \exp[bU_{t-1}^F]/(\exp\left[bU_{t-1}^C\right] + \exp[bU_{t-1}^F])$, where U_t^C and U_t^F stand for the fitness of the extrapolative and regressive expectation rule, measured by past realized profits, and parameter $b > 0$ denotes the speculators' intensity of choice. Past realized profits of the extrapolative rule are given by $U_t^C = (X_t - (1+r)X_{t-1})D_{t-1}^C$ and depend on the realized excess return per share of the risky asset over the risk-free asset, i.e. $(X_t - (1+r)X_{t-1})$, and the demand for the risky asset suggested by this rule, i.e. $D_{t-1}^C = (E_{t-1}^C[X_t] - (1+r)X_{t-1})/c$ with $c > 0$. Similarly, past realized profits of the regressive rule result in $U_t^F = (X_t - (1+r)X_{t-1})\, D_{t-1}^F - d$, where $D_{t-1}^F = (E_{t-1}^F[X_t] - (1+r)X_{t-1})/c$ represents the demand according to the regressive rule and $d \geq 0$ reflects a possible per period cost associated with forming regressive expectations.

Figure 9 contains two scenarios for which the model by Brock and Hommes (1998) generates a bimodal distributed distortion (since the model is formulated in deviations from the fundamental value, distortion is measured by the distance between prices and fundamental values). The top left panel of Figure 9 depicts a simulation run with 2,000 observations for $a = 1.2$, $b = 2.5$, $c = 1$, $d = 1$, $r = 0.1$ and $\sigma = 0.05$. This parameter setting implies that the deterministic skeleton of the model possesses two locally stable steady states, one located in the bull market region and the other located in the bear market region. Due to exogenous shocks, erratic switches between bull and bear market dynamics emerge.[23] The top right panel of Figure 9 reveals that the associated distribution of the distortion is bimodal. The bottom left panel of Figure 9 contains the development of X_t for 800 time steps, assuming that $a = 1.2$, $b = 3.6$, $c = 1$, $d = 1$, $r = 0.105$ and $\sigma = 0.01$. In the absence of exogenous shocks, this parameter setting gives rise to two locally stable limit cycles. One limit cycle implies fluctuations above the fundamental value while the other limit cycle implies fluctuations below the fundamental value. In the presence of exogenous shocks, the dynamics switches between the two (noisy) attractors. As can be seen from the bottom right panel of Figure 9, these forces yield a bimodal distributed distortion.

*** Figure 9 about here ***

7 Conclusions

Shiller (2015) makes it clear that stock markets are prone to bubbles and crashes but, fortunately, track their fundamental values in the long run. In this paper, we define a stock market's distortion as the log difference between its price level and its fundamental value. One question arising in this context, then, is what the distribution of a stock market's distortion might look like. Having a

[23] Recall that such an explanation is also offered by Zeeman (1974) and Lux (1995). The difference between the three models is that they imply different adjustment speeds towards the bull and bear market steady states. While the adjustment in the model by Zeeman (1974) is very fast (instantaneous), the adjustment in the model by Brock and Hommes (1998) is rather slow. The dynamics of the Lux (1995) model is somewhere in between.

standard linear time series model in mind, most economists would presumably predict that the distribution of a stock market's distortion is unimodal (and bell-shaped). The key empirical message of our paper is thus of a rather puzzling nature: we show that the distribution of the S&P 500's distortion is bimodal, i.e. the S&P 500 spends relatively more time in bull and bear markets than in the vicinity of its fundamental value. The main theoretical insight offered by our paper is that we are able to resolve this puzzle. Agent-based financial market models, such as the ones by Gaunersdorfer and Hommes (2007) and Franke and Westerhoff (2012), explain the dynamics of stock markets by nonlinear interactions of speculators relying on technical and fundamental prediction rules. Close to the fundamental value, stock markets are dominated by the destabilizing trading behavior of chartists and, consequently, stock prices are quickly pushed away from fundamental values. As bull or bear markets become more pronounced, however, speculators place more weight on fundamental analysis. Given that the mean reversion pressure exercised by fundamentalists is rather weak, it takes a while for a stock market's distortion to normalize again. We consider it worth mentioning that the bimodality of the distribution of the S&P 500's distortion confirms an implicit prediction of a number of agent-based financial market models, including the seminal contributions by Zeeman (1974), Day and Huang (1990), Chiarella (1992), de Grauwe et al. (1993), Lux (1995) and Brock and Hommes (1998).

We conclude our paper by briefly pointing out three avenues for future research. First, our empirical analysis rests on Shiller's (2015) unique S&P 500 dataset. Of course, it would be interesting to extend our study to other speculative markets. Besides stock markets, one may also consider foreign exchange markets, commodity markets or housing markets. A major challenge in this endeavor is that one needs sufficiently long time series. Second, we show that existing agent-based financial market models are able to explain the bimodal shape of the distribution of the distortion of the S&P 500. Alternatively, one may develop a novel agent-based financial market model to mimic the stylized facts of the S&P 500. The calibration or estimation of such a model can then explicitly take into account the properties of the S&P 500's distortion, e.g. the shape of its distribution or the average duration of bull and bear markets. The method of simulated moments, applied to estimate the model by Franke and Westerhoff (2012), may be quite powerful in this respect. By quantifying the usual stylized facts plus some distortion properties, the method of simulated moments also allows us to rank different agent-based financial market models in their ability to explain the behavior of the S&P 500.[24] Third, our theoretical analysis focuses on agent-based financial market models. It seems worthwhile to explore whether there are other economic reasons for the distributional properties of the S&P 500's distortion. Our contribution poses a first benchmark in this respect. To sum up, we provide empirical evidence and theoretical explanations for the puzzling bimodality of the distribution of the S&P 500's distortion and hope that our paper stimulates more work in this exciting research direction.

[24] As remarked by an anonymous referee, the average duration of bull and bear markets produced by the models by Gaunersdorfer and Hommes (2007) and Franke and Westerhoff (2012) seems to be too short. Nevertheless, it is interesting to see how well both models replicate crucial properties of the S&P 500's distortion. One can find out which of the two models does a better job in this respect by re-estimating these approaches using the method of simulated moments.

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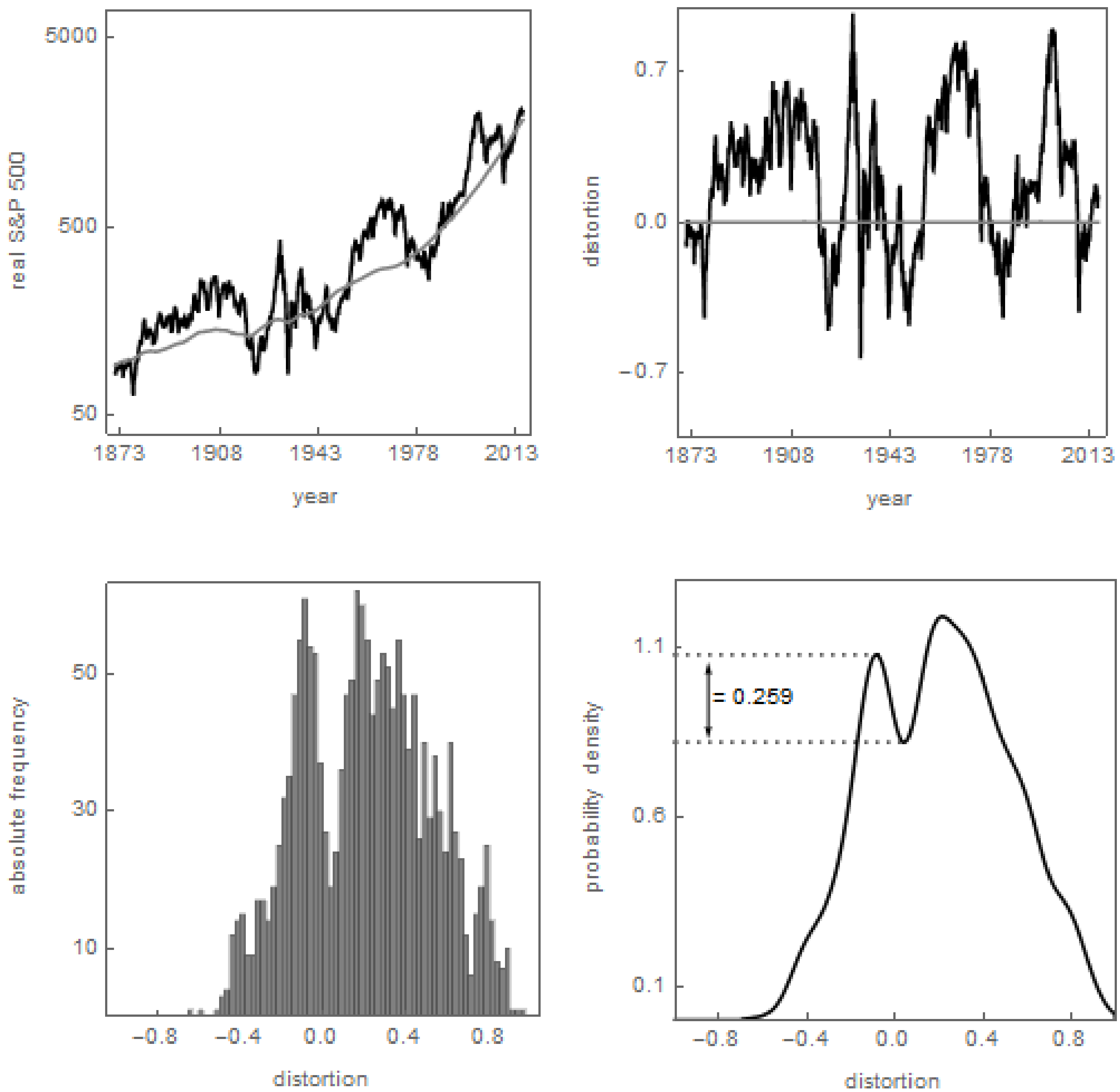

Figure 1: Properties of the S&P 500. The top left panel shows the evolution of the real S&P 500 (black line) and its real fundamental value (gray line) between 1871 and 2015. The top right panel shows the log difference between these two time series. The bottom left panel shows a histogram of the S&P 500's distortion. The bottom right panel shows the same as the bottom left panel, except that the histogram is smoothed. The underlying data and the computation of the fundamental value are from Shiller (2015).

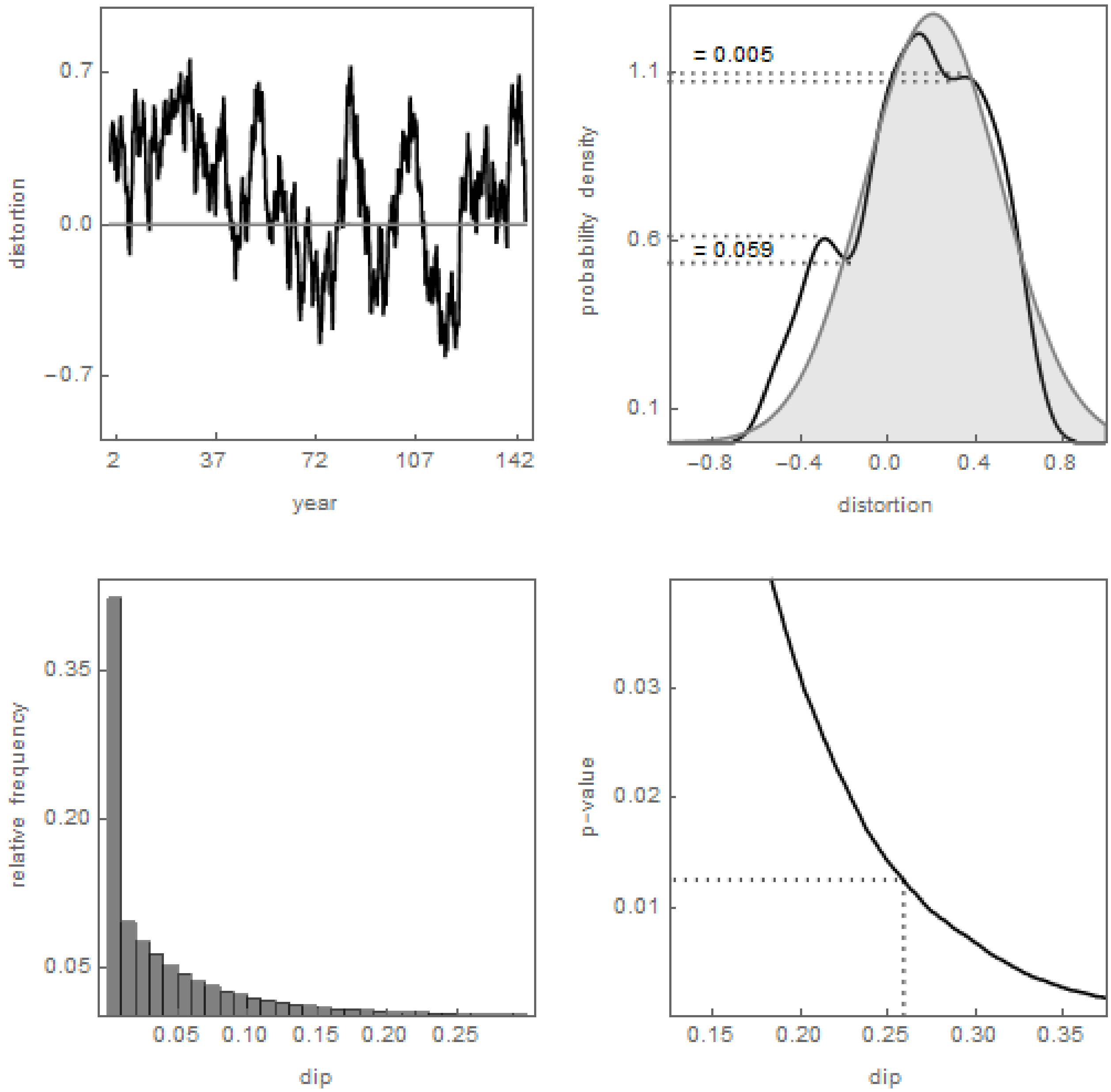

Figure 2: Properties of the estimated ARMA (2,2) model. The top left panel shows a simulation run of the estimated ARMA (2,2) model for a time span of 145 years. The top right panel shows a smoothed distribution for this simulation run (black line) along with its asymptotic distribution (gray shaded area). The bottom left panel shows the relative frequency of observed dips computed from 100,000 simulation runs of the estimated ARMA (2,2) model. The bottom right panel shows the corresponding *p*-values.

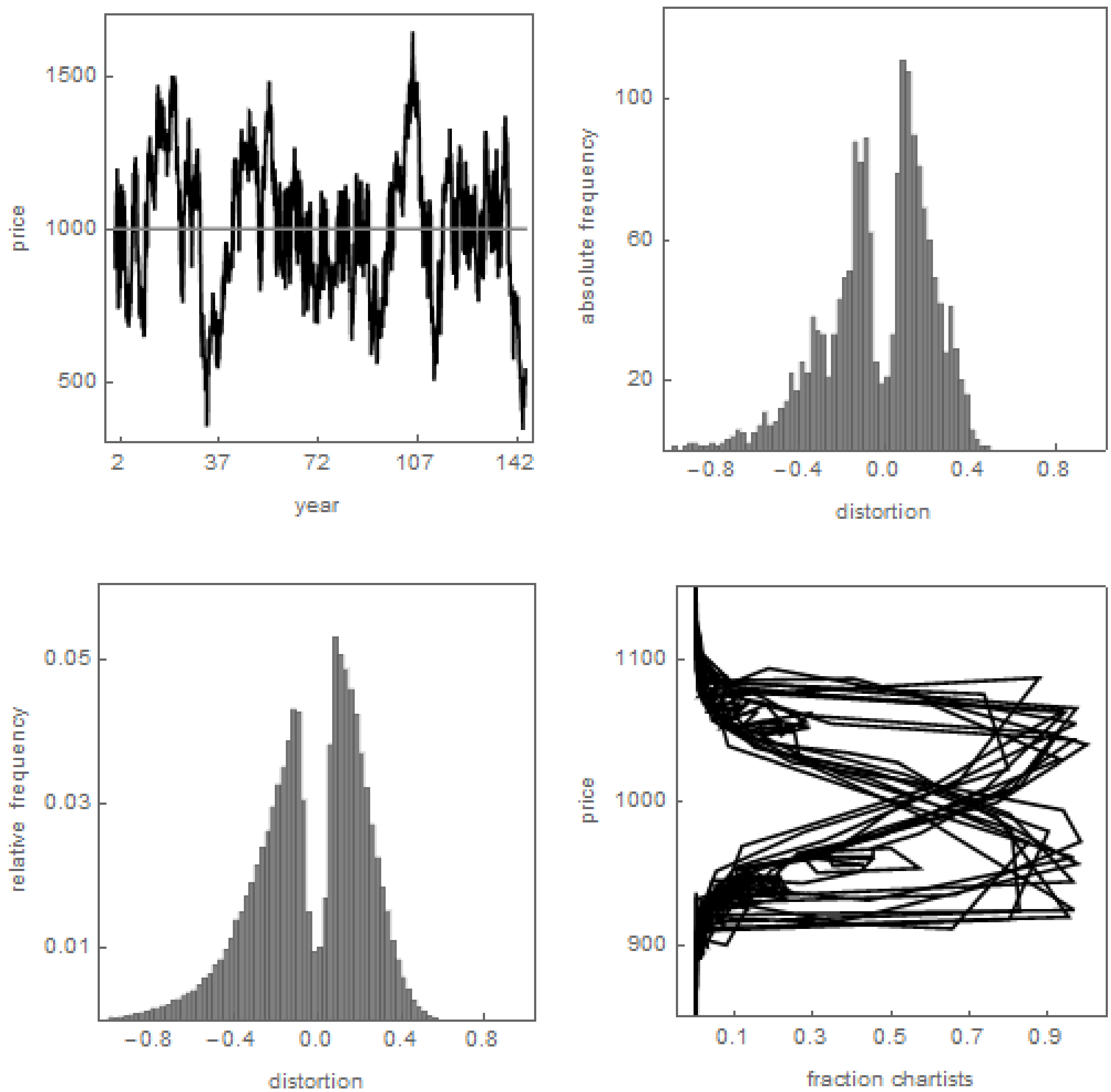

Figure 3: Properties of the stochastic model by Gaunersdorfer and Hommes. The top left panel shows the evolution of the price (black line) and its fundamental value (gray line) for a time span of 145 years (to obtain $12 \cdot 145 = 1{,}740$ monthly observations, every 21^{st} observation out of $21 \cdot 12 \cdot 145 = 36{,}540$ daily observations is plotted). The top right panel shows a histogram of the distortion of this simulation run. The bottom left panel shows the asymptotic distribution of the model's distortion. The bottom right panel shows the price versus the market share of chartists for a shorter time window. Model parameters are taken from Gaunersdorfer and Hommes (2007) and reported in Section 4.

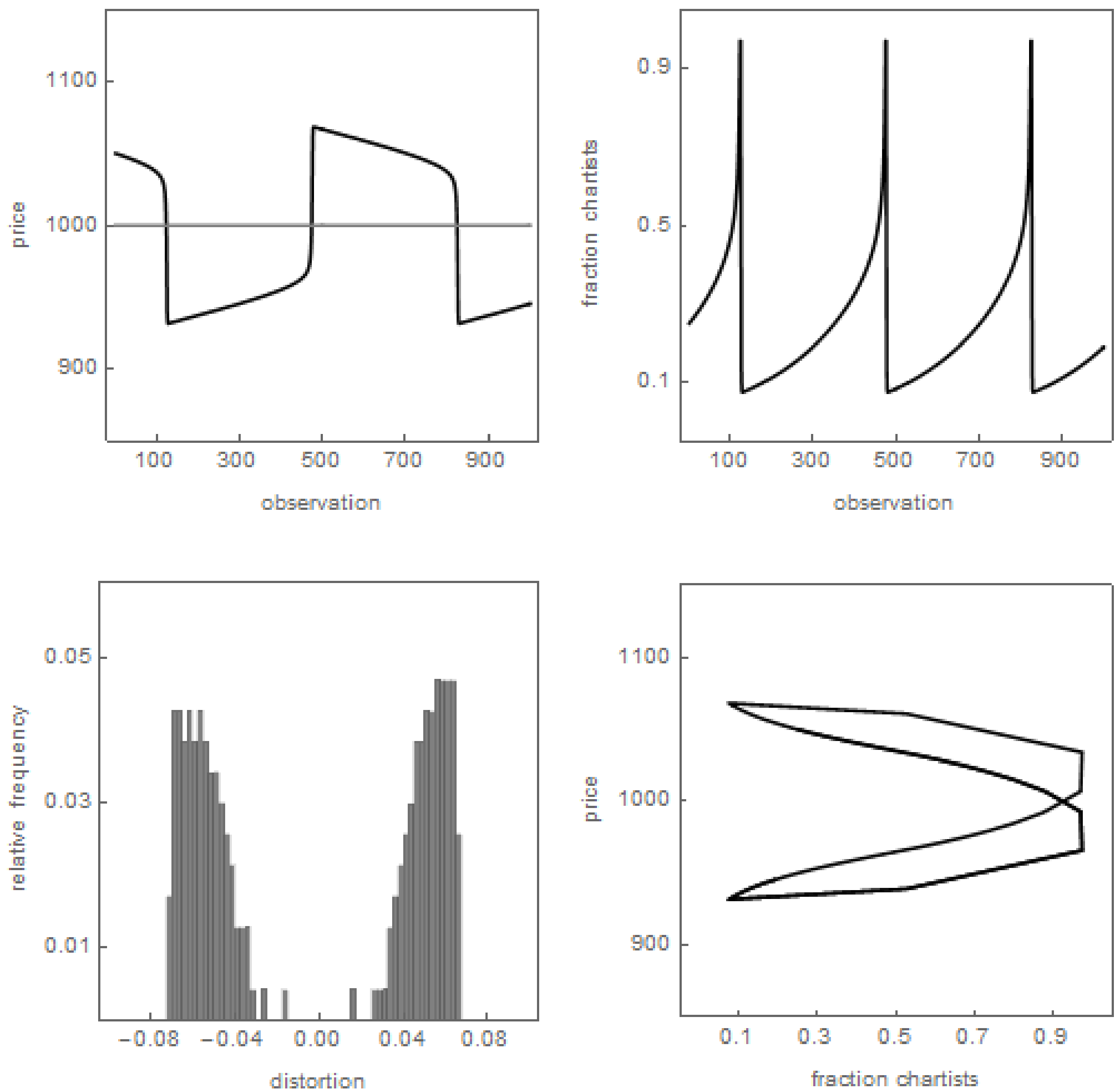

Figure 4: Properties of the deterministic model by Gaunersdorfer and Hommes. The top left panel shows the evolution of the price (black line) and its fundamental value (gray line) for 1,000 observations. The top right panel shows the corresponding market shares of chartists. The bottom left panel shows the asymptotic distribution of the model's distortion. The bottom right panel shows the price versus the market share of chartists. Model parameters are taken from Gaunersdorfer and Hommes (2007) and reported in Section 4.

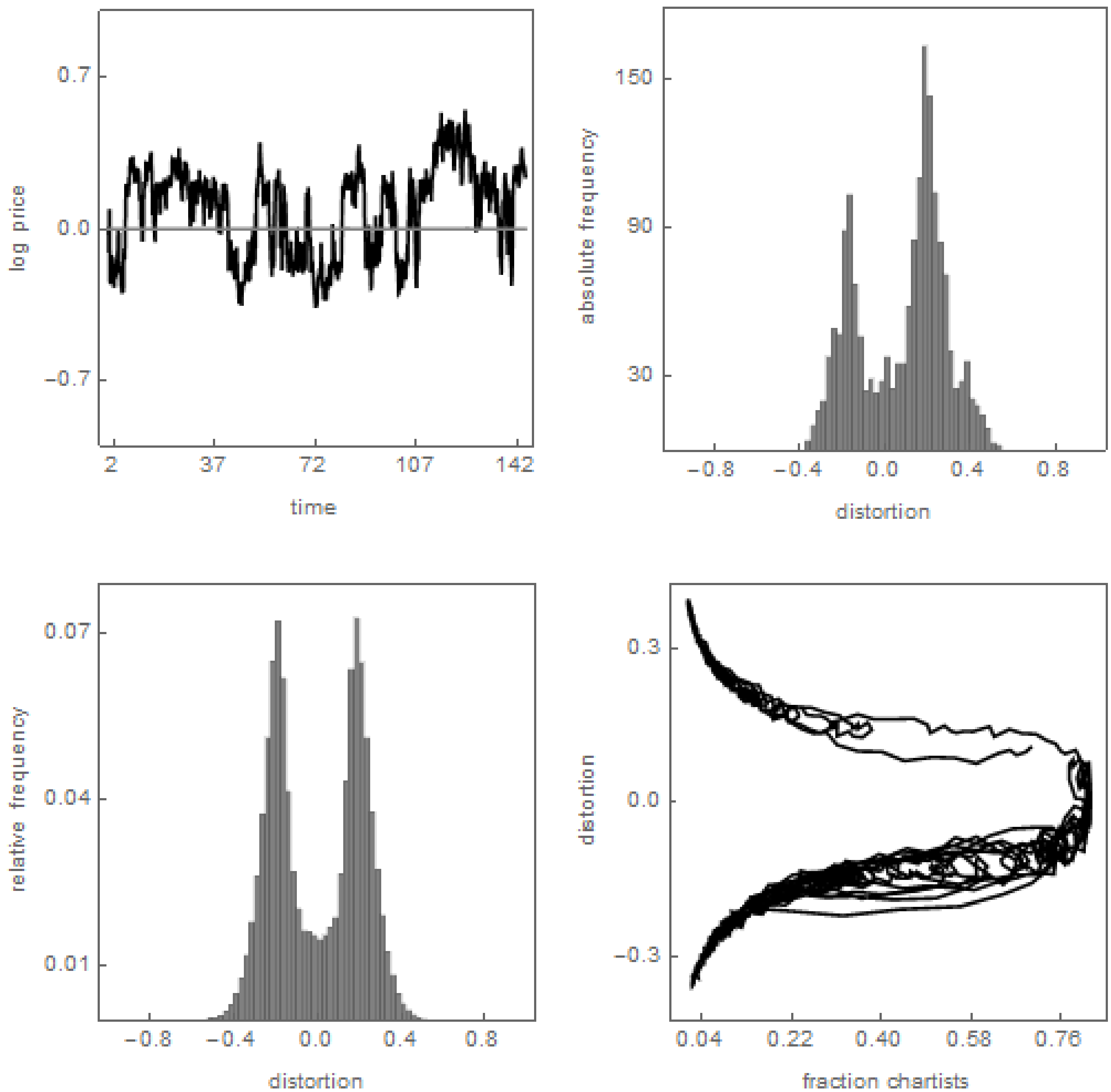

Figure 5: Properties of the stochastic model by Franke and Westerhoff. The top left panel shows the evolution of the log price (black line) and its log fundamental value (gray line) for a time span of 145 years (to obtain $12 \cdot 145 = 1{,}740$ monthly observations, every 21st observation out of $21 \cdot 12 \cdot 145 = 36{,}540$ daily observations is plotted). The top right panel shows a histogram of the distortion of this simulation run. The bottom left panel shows the asymptotic distribution of the model's distortion. The bottom right panel shows the price versus the market share of chartists for a shorter time window. Model parameters are taken from Franke and Westerhoff (2012) and reported in Section 5.

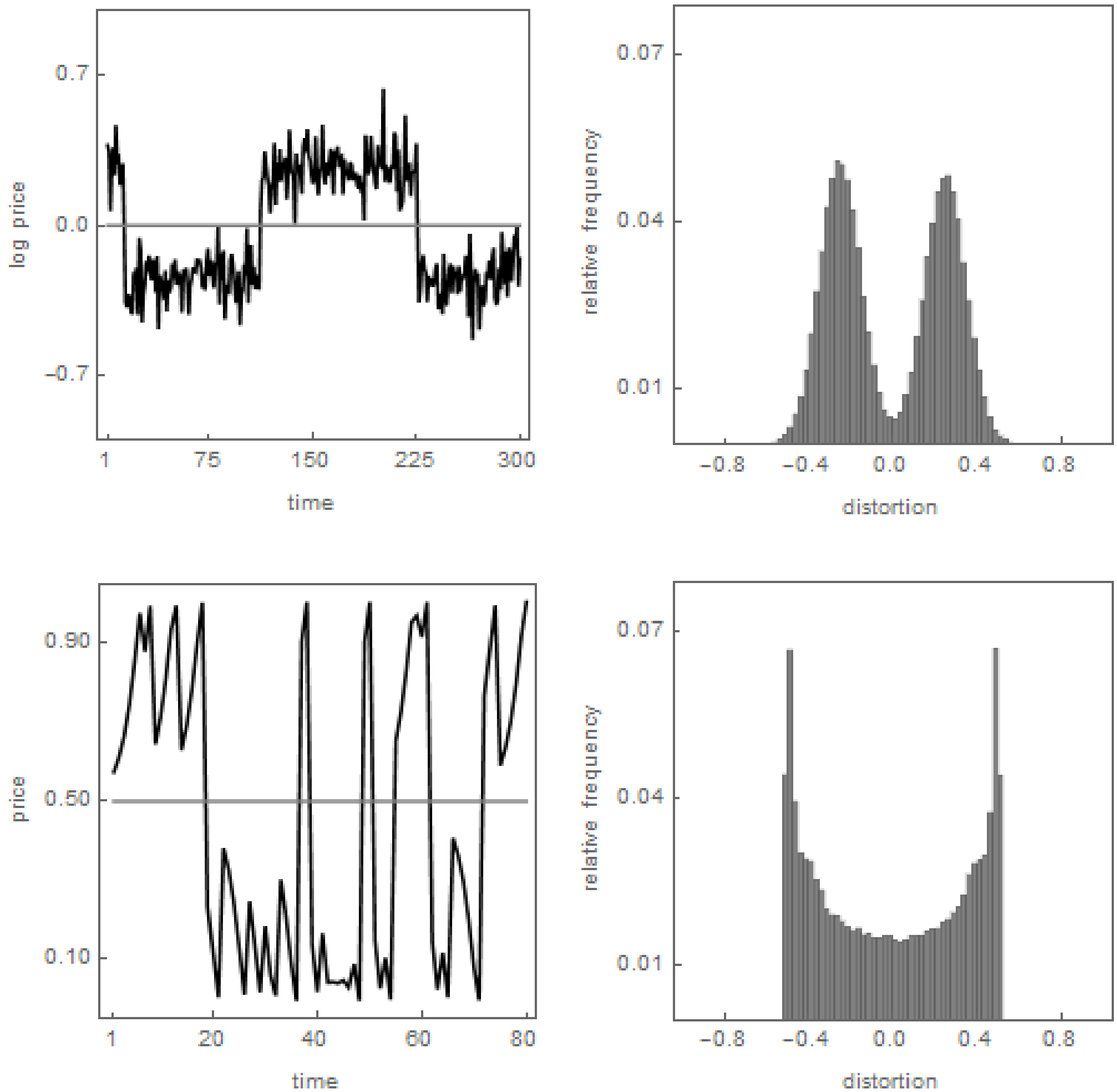

Figure 6: Properties of the models by Zeeman and Day and Huang. The top left panel shows the evolution of the log price (black line) and its log fundamental value (gray line) for the model by Zeeman (1974). The top right panel shows the corresponding distribution of the distortion. The bottom left panel shows the evolution of the price (black line) and its fundamental value (gray line) for the model by Day and Huang (1990). The bottom right panel shows the corresponding distribution of the distortion. Model parameters are reported in Section 6.

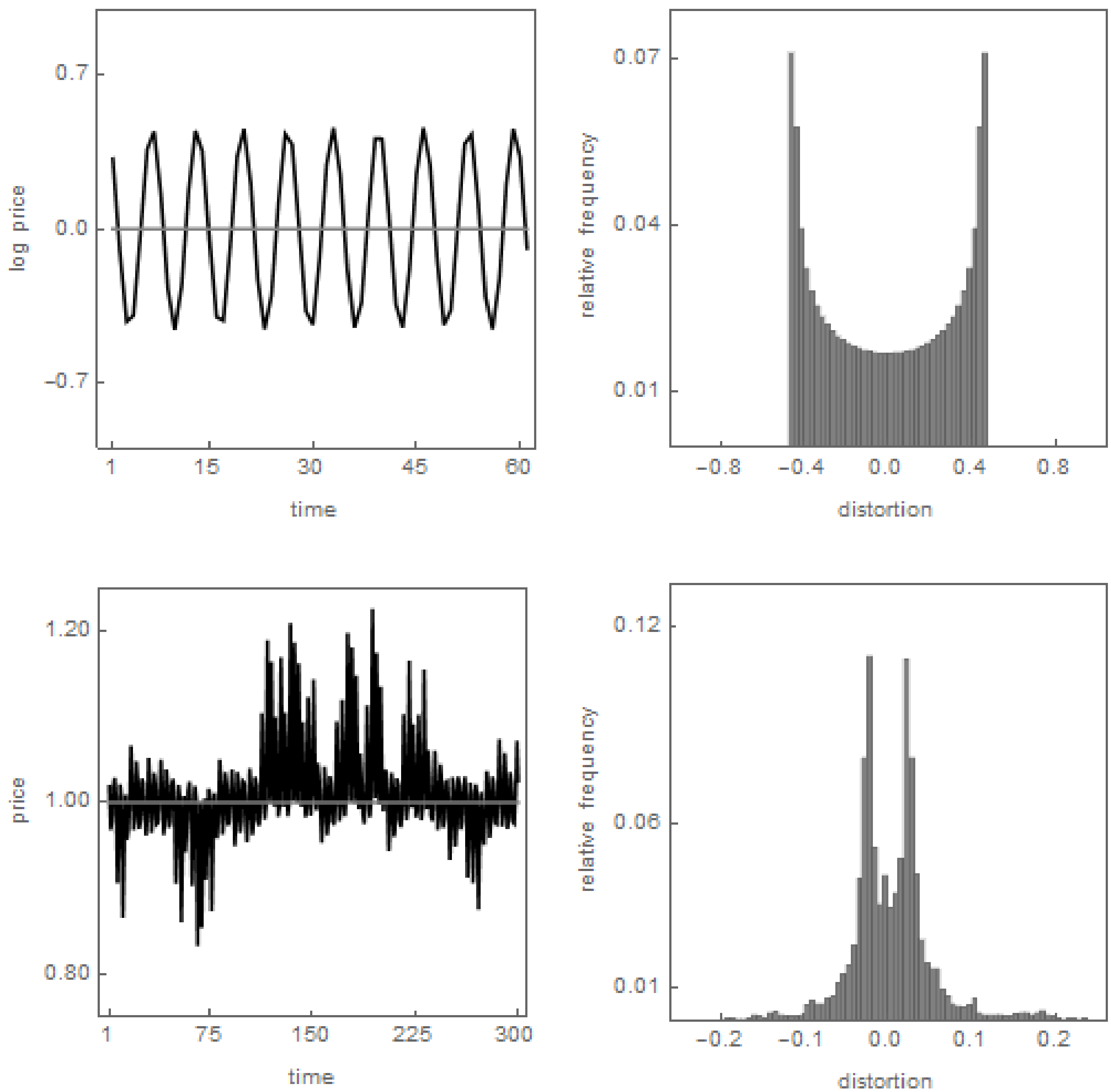

Figure 7: Properties of the models by Chiarella and de Grauwe et al. The top left panel shows the evolution of the log price (black line) and its log fundamental value (gray line) for the model by Chiarella (1992). The top right panel shows the corresponding distribution of the distortion. The bottom left panel shows the evolution of the price (black line) and its fundamental value (gray line) for the model by de Grauwe et al. (1993). The bottom right panel shows the corresponding distribution of the distortion. Model parameters are reported in Section 6.

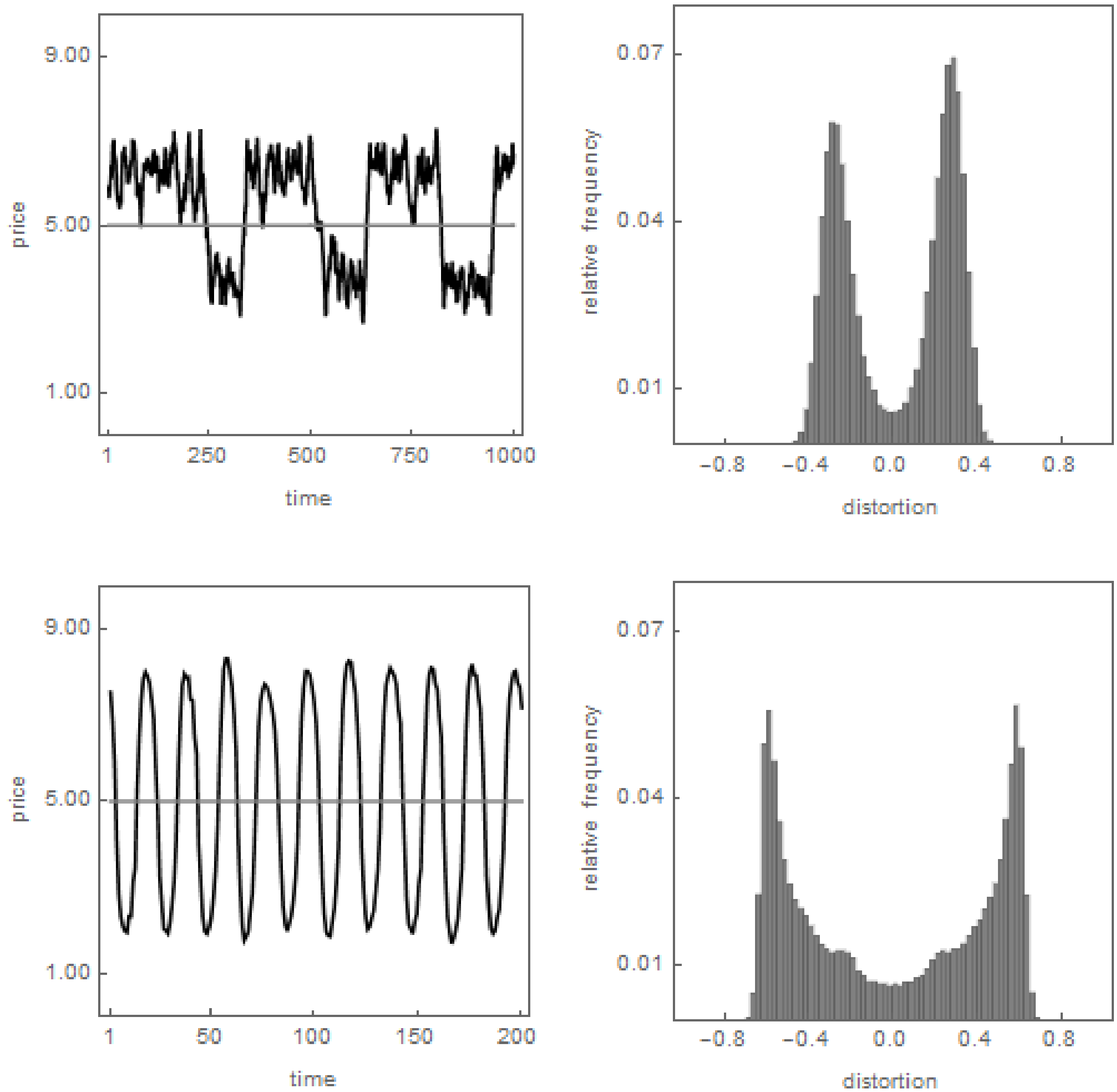

Figure 8: Properties of the model by Lux. The top left panel shows the evolution of the price (black line) and its fundamental value (gray line) for a parameter setting in which the deterministic model by Lux (1995) possesses locally stable bull and bear market steady states. The top right panel shows the corresponding distribution of the distortion. The bottom panels show the same except that the parameter setting now gives rise to a limit cycle, subject to some exogenous noise. Model parameters are reported in Section 6.

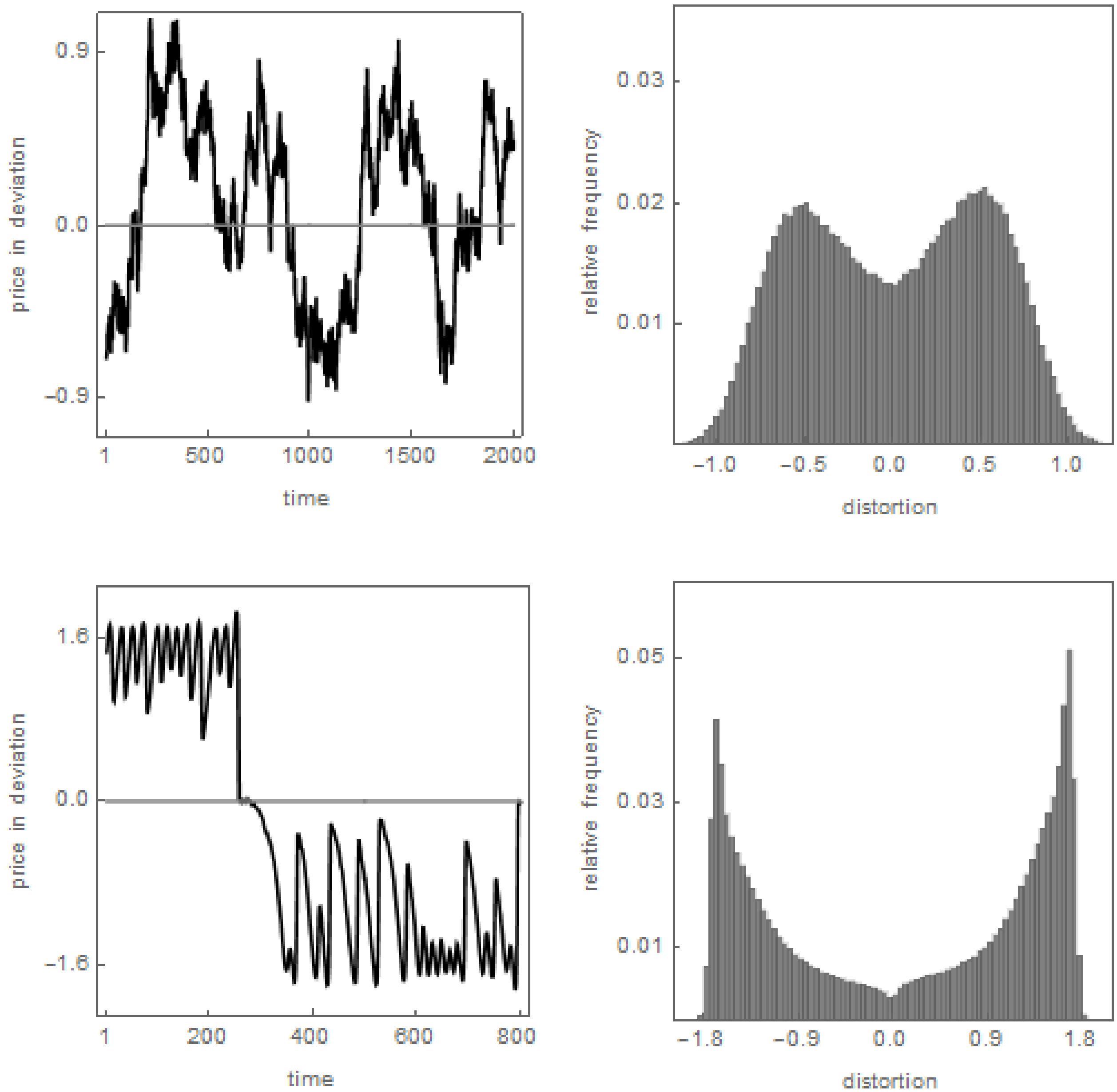

Figure 9: Properties of the model by Brock and Hommes. The top left panel shows the evolution of the price in deviation from its fundamental value (black line) for a parameter setting in which the deterministic model by Brock and Hommes (1998) possesses locally stable bull and bear market steady states. The top right panel shows the corresponding distribution of the distortion. The bottom panels show the same except that the parameter setting now gives rise to two coexisting locally stable limit cycles, subject to some exogenous noise. Model parameters are reported in Section 6.

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